On Julia sets for quasimeromorphic mappings

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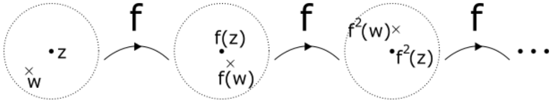
Fatou set $\mathcal{F}(f)$ and Julia set $\mathcal{J}(f)$

Study the behaviour of sequences of iterates $(f^n(z))_{n=0}^{\infty}$ for different starting values of z.

Recall (f analytic or mero):

$\mathcal{F}(f) := \{z \in \hat{\mathbb{C}} : \{f^n\}$ is equicontinuous on a nbhd of $z\}$;

$\mathcal{J}(f) := \hat{\mathbb{C}} \setminus \mathcal{F}(f)$.



$\mathcal{F}(f)$ is ‘regular’, while $\mathcal{J}(f)$ is ‘chaotic’.

Example and properties

$f : \mathbb{C} \to \hat{\mathbb{C}}$ trans mero, $\text{card}(\mathcal{O}_f^-(\infty)) \geq 3$

$\implies \mathcal{O}_f^-(\infty)$ is infinite (Picard theorem),

$\implies \hat{\mathbb{C}} \setminus \overline{\mathcal{O}_f^-(\infty)}$ is largest open set where $\{f^n\}$ defined,

$\implies \mathcal{F}(f) = \hat{\mathbb{C}} \setminus \overline{\mathcal{O}_f^-(\infty)}$ and $\mathcal{J}(f) = \overline{\mathcal{O}_f^-(\infty)}$ (Montel theorem).

Some classic $\mathcal{J}(f)$ properties (f mero):

Non-empty, closed, infinite, perfect, completely invariant;

'Blowing-up' property: U open, $U \cap \mathcal{J}(f) \neq \varnothing \implies |\hat{\mathbb{C}} \setminus \mathcal{O}_f^+(U)| \leq 2$.

Qr and qm mappings

A continuous map $f : \mathbb{R}^d \to \mathbb{R}^d$ is quasiregular (qr) if $f \in W^1_{d,loc}(\mathbb{R}^d)$, and there is some $K_I \geq 1$ s.t.

$$\|Df(x)\|^d \leq K_I J_f(x) \text{ a.e.}$$

For $K \geq 1$, f is called K-qr if the amount of local stretching is uniformly bounded by K.

$g : \mathbb{R}^d \to \hat{\mathbb{R}}^d$ is quasimeromorphic (qm) of trans type if g or $M \circ g$ is locally qr, with $M : \hat{\mathbb{R}}^d \to \hat{\mathbb{R}}^d$ a sense-preserving Möbius transformation s.t. $M(\infty) \neq \infty$.

If g is qr and f is qr (qm), then $f \circ g$ is qr (qm) with

$$K(f \circ g) \leq K(f)K(g).$$

Examples and properties of qr and qm

- Analytic functions are 1-qr; meromorphic functions are 1-qm, quasiconformal mappings are injective qr mappings.
- (Zorich, '67) *Zorich map* (analogue of exponential function) $Z : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ is qr.
- (Martio, Srebro, '75) $p_d : \mathbb{R}^d \to \hat{\mathbb{R}}^d$ is a d-periodic qm map of trans type with an infinite number of poles.

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Non-constant qr mappings are open, discrete and differentiable a.e.

Rickman proved analogue of Picard theorem for qm.

Theorem (Rickman, '80)

Let $d \geq 2$, $K \geq 1$. Then there exists a constant $q = q(d, K)$ s.t.

(i) every qm map $f : \mathbb{R}^d \to \hat{\mathbb{R}}^d \setminus \{a_1, a_2, \ldots, a_q\}$ is constant;

(ii) if $g : \mathbb{R}^d \to \hat{\mathbb{R}}^d$ is qm of trans type, then $g^{-1}(\{a_1, a_2, \ldots, a_q\})$ is infinite.

Fatou and Julia sets for qr and qm?

We can iterate qr and qm mappings (when away from poles).

If f^n is K-qr for all n, then can use Fatou definition directly - uniformly quasiregular (uqr) mappings.

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For a general K-qr map f, f^n can be K^n-qr... this blows up as n gets large!

Conclusion: Can't define the Julia set as the complement of the Fatou set for general qr and qm maps.

Solution (Sun, Yang, '99 -'01; Bergweiler, '13; Bergweiler, Nicks, '14)

For f K-qr with $K < \deg(f)$, define the Julia set using a version of the 'blowing-up' property.

Julia sets for qr in higher dimensions

For a K-qr mapping f with $K < deg(f)$, define

$$\mathcal{J}_{\mathsf{FINITE}}(f) = \{x : \hat{\mathbb{R}}^d \setminus \mathcal{O}_f^+(U) \text{ is } \textbf{finite} \text{ for all nbhds } U \text{ of } x\};$$
$$\mathcal{J}_{\mathsf{SMALL}}(f) = \{x : \hat{\mathbb{R}}^d \setminus \mathcal{O}_f^+(U) \text{ is } \textbf{`small'} \text{ for all nbhds } U \text{ of } x\}.$$

'small' refers to (conformal) capacity zero, and $\mathcal{J}_{\mathsf{FINITE}}(f) \subset \mathcal{J}_{\mathsf{SMALL}}(f)$.

When $\mathcal{J}_{\mathsf{FINITE}}(f)$ or $\mathcal{J}_{\mathsf{SMALL}}(f)$ are non-empty, they satisfy some similar properties to their classical $\mathcal{J}(f)$ counterpart.

The 'small' condition is used since we are unable to prove that $\mathcal{J}_{\mathsf{FINITE}}(f) \neq \varnothing$ for some types of qr mapping.

Examples of qr Julia sets studied, by type

$\mathcal{J}_{\text{SMALL}}(f)$: (Bergweiler, '13) $f : \hat{\mathbb{R}}^d \to \hat{\mathbb{R}}^d$ qr with finite degree (**);
(Bergweiler, Nicks, '14) $f : \mathbb{R}^d \to \mathbb{R}^d$ qr of trans type, (**).

$\mathcal{J}_{\text{FINITE}}(f)$: (Sun, Yang, '99 - '01) $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ qr with finite degree.
(Bergweiler, Nicks, '14) $f : \mathbb{C} \to \mathbb{C}$ qr of trans type.
(Nicks, Sixsmith, (to appear)) $f : \hat{\mathbb{R}}^d \setminus S \to \hat{\mathbb{R}}^d \setminus S$ qr with $S \ni \infty$, $\text{card}(S) \geq 2$, a set of isolated essential singularities.
Denote this Julia set as $\mathcal{J}_S(f)$.

Open conjecture

For f qr, $\mathcal{J}_{\text{SMALL}}(f) = \mathcal{J}_{\text{FINITE}}(f)$.

**: extra condition on f needed for full analogous properties

Deeper look into qm trans case

Let $f : \mathbb{R}^d \to \hat{\mathbb{R}}^d$ be K-qm of trans type.

Obsv: No poles $\implies$ $f : \mathbb{R}^d \to \mathbb{R}^d$ is qr of trans type.

By Rickman's theorem (ii), 2 other cases arise:

(1) $2 \leq \text{card}(\mathcal{O}_f^-(\infty)) < q$;

(2) $\mathcal{O}_f^-(\infty)$ is infinite.

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Case (1): Consider $f^q : \hat{\mathbb{R}}^d \setminus S \to \hat{\mathbb{R}}^d \setminus S$, which is qr with $S := \mathcal{O}_f^-(\infty)$ a set of isolated essential singularities. Here, $\mathcal{J}_S(f^q)$ is defined. Thus

$$(1) \text{ suggests } \mathcal{J}(f) := \mathcal{J}_S(f^q) \cup S.$$

Case (2): Consider equivalent mero case, where $\mathcal{J}_{mero}(g) = \overline{\mathcal{O}_g^-(\infty)}$. Thus

$$(2) \text{ suggests } \mathcal{J}(f) := \overline{\mathcal{O}_f^-(\infty)}.$$

Julia set for qm of trans type

Combining (1) and (2) gives the following definition.

Definition (W., '18)

For $f : \mathbb{R}^d \to \hat{\mathbb{R}}^d$ qm of trans type with at least 1 pole, the Julia set of f is given by

$$\begin{aligned}\mathcal{J}(f) = &\{x \in \hat{\mathbb{R}}^d \setminus \overline{\mathcal{O}_f^-(\infty)} : \hat{\mathbb{R}}^d \setminus \mathcal{O}_f^+(U) \text{ is } \textbf{finite} \text{ for} \\ &\text{all nbhds } U \subset \hat{\mathbb{R}}^d \setminus \overline{\mathcal{O}_f^-(\infty)} \text{ of } x\} \cup \overline{\mathcal{O}_f^-(\infty)}.\end{aligned}$$

Results about $\mathcal{J}(f)$

Theorem (W., '18)

(a) For $f : \mathbb{C} \to \hat{\mathbb{C}}$ trans mero with at least 1 pole, $\mathcal{J}(f)$ agrees with the usual one.

(b) $\mathcal{J}(f)$ is non-empty, closed, infinite and perfect.

(c) $x \in \mathcal{J}(f) \setminus \{\infty\}$ if and only if $f(x) \in \mathcal{J}(f)$.

(d) $\mathcal{J}(f)$ intersects the escaping set $I(f)$, the bounded orbit set $BO(f)$ and the bungee set $BU(f)$.

(e) $\mathcal{J}(f) \subset \partial I(f) \cap \partial BO(f) \cap \partial BU(f)$, but examples of qm maps exist with strict inclusion.

As $\mathcal{J}(f)$ uses the finiteness condition, this provides support for the open conjecture.

Further questions

- Examples of higher dimensional qm of trans type with $\mathcal{O}_f^-(\infty)$ finite?
- Results on $\mathcal{QF}(f) := \hat{\mathbb{R}}^d \setminus \mathcal{J}(f)$?
- Structure results for $\mathcal{J}(f)$?
- Define $\mathcal{J}(f)$ for a broader class of mappings (e.g. analogue of Bolsch $\mathcal{K}$ class for qm)?