Baker's conjecture in complex dynamics

f is a transcendental entire function

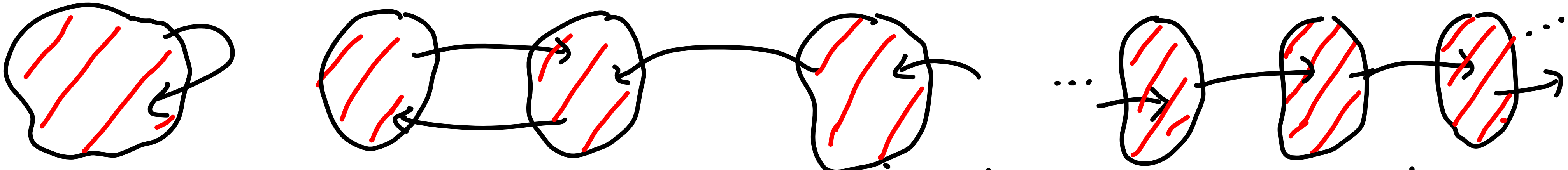
Joint work with Gwyneth Stallard

Fatou set

$$F(f) = \{ z \in \mathbb{C} : (f^n) \text{ is equicontinuous near } z \}$$ stable set

Julia set $J(f) = \mathbb{C} \setminus F(f) \neq \emptyset$ chaotic set

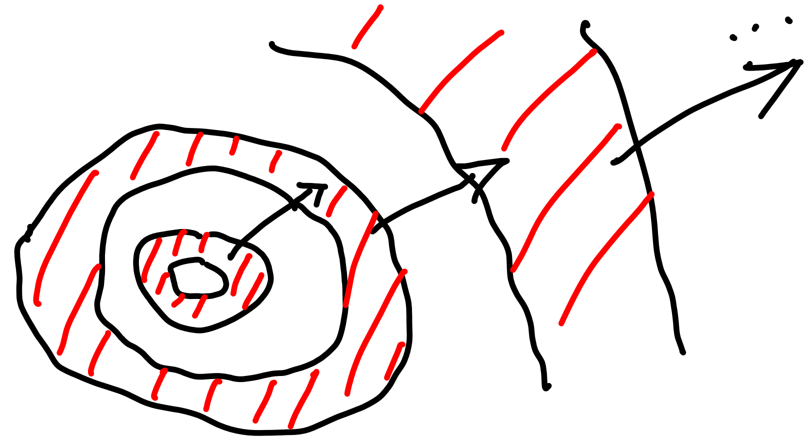
Fatou components (bounded or unbounded)



Examples of wandering domains

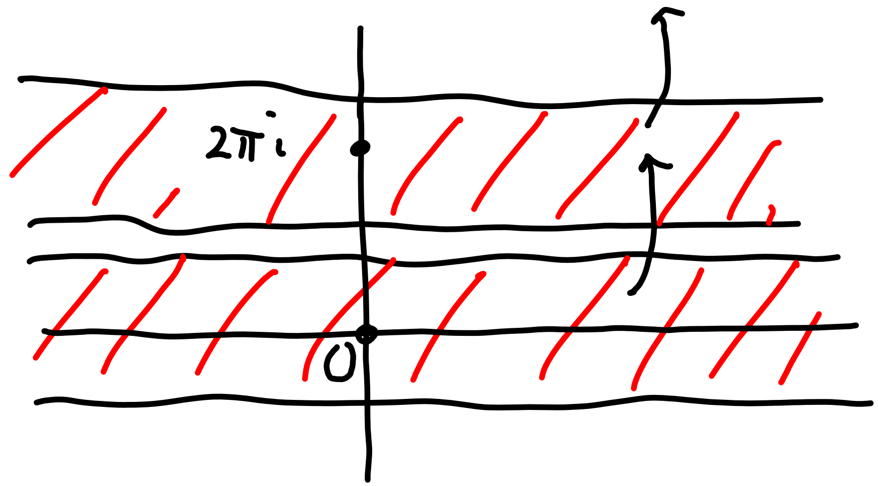
Baker 1976

$$f(z) = z^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)$$



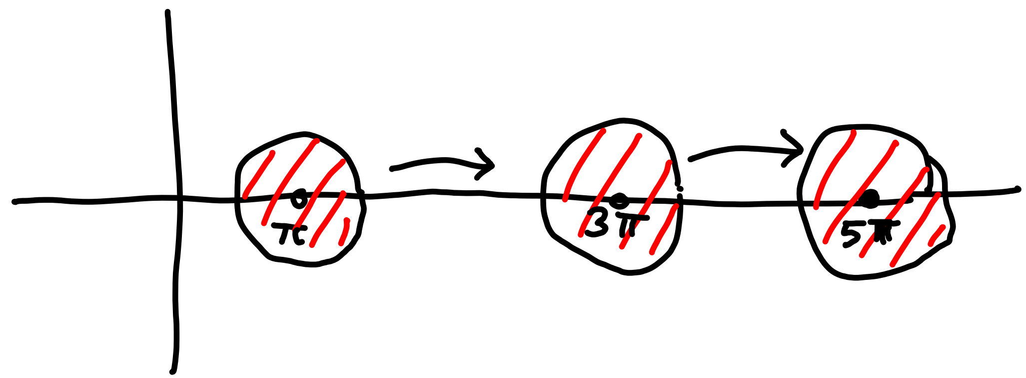
Herman 1982

$$f(z) = z + e^{-z} - 1 + 2\pi i$$



Devaney 198?

$$f(z) = z + \sin z + 2\pi$$



$\boxed{\text{Baker's conjecture } (\sim 1981) \text{ If } f \text{ has order} < \tfrac{1}{2}, \text{ then } f \text{ has no unbounded Fatou components.}}$

maybe even order $\frac{1}{2}$, min type

$\boxed{\text{Eremenko's conjecture (1989) All components of } I(f) \text{ are unbounded.}}$

Order of f: $\quad \rho = \rho(f) = \overline{\lim_{r \to \infty}} \dfrac{\log \log M(r)}{\log r}$,

$$M(r) = \max_{|z|=r} |f(z)| \qquad m(r) = \min_{|z|=r} |f(z)|.$$

Baker 1980/81

1. $\log M(r) = O((\log r)^p)$, $0 < p < 3$ $\Rightarrow$ no unbdd Fatou cmpnts

2. $\log M(r) = o(r^{1/2})$ $\Rightarrow$ no unbdd invariant Fatou cmpnts.

Early progress

Stallard 1993 Solved conjecture when

1. $\log\log M(r) \leq \dfrac{(\log r)^{1/2}}{(\log\log r)^{\varepsilon}}$, $\varepsilon > 0$

2. $\rho < \frac{1}{2}$ and $\displaystyle\lim_{r\to\infty} \frac{\log M(2r)}{\log M(r)}$ exists

Anderson + Hinkkanen 1998 Solved conjecture when $\rho < \frac{1}{2}$ and

$\exists c > 0$ s.t. $\forall k > 1$ $M(r^k) \geq M(r)^{dk}$, r large,

where $d = k^c > 1$.

Further progress using minimum modulus/stretching

Zheng 2000

$\log M(r) = o(r^{\frac{1}{2}}) \Rightarrow$ no unbdd periodic Fatou cmpnts

Various: $\rho < \frac{1}{2}$ + weaker regularity $\Rightarrow$ no unbdd Fatou cmpnts

R+S 2009 For some $m \geq 2$,

$$\frac{\log \log M(r)}{\log r} = \frac{O(1)}{\log^m r} \Rightarrow \text{no unbdd Fatou cmpnts}$$

Similar result by Hinkkanen + Miles 2009.

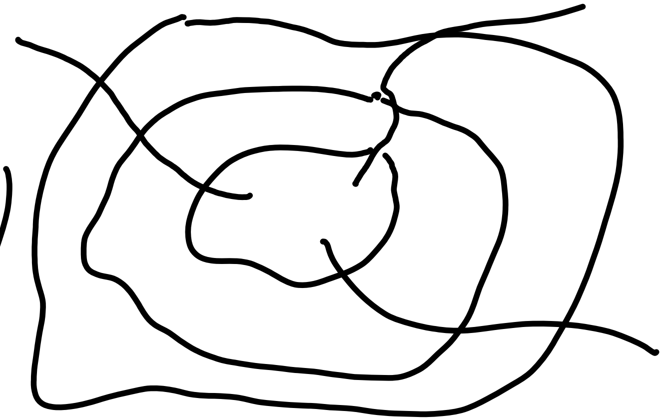
Observation Stretching proofs show $A_R(f)$ is a spider's web.

Role of spiders' webs

Recall: for R s.t. $M^n(R) \to \infty$, define

$$A_R(f) = \{ z : |f^n(z)| \geq M^n(R),\ n = 1, 2, \cdots \}.$$

If $A_R(f)^c$ has no unbdd components, then $A_R(f)$ is a spider's web.



$\Rightarrow$ $A_R(f)$ contains arb. large loops in $J(f)$

$\Rightarrow$ no unbdd Fatou components!

$\boxed{\text{Question: Does } \rho < \tfrac{1}{2} \text{ imply } A_R(f) \text{ is a spider's web?}}$

Minimum modulus & stretching – reaching the limit

R+S 2012 Let

$$R_n = M^n(R) \to \infty \quad \text{and} \quad \varepsilon_n = \max_{R_n \le r \le R_{n+1}} \frac{\log\log M(r)}{\log r}.$$

Then

$$(*) \qquad \sum_{n=1}^{\infty} \varepsilon_n < \infty \Rightarrow A_R(f) \text{ is a spider's web.}$$

$\Rightarrow$ no unbdd Fatou cmpnts

In a sense $(*)$ is sharp – in particular, $\exists$ functions of order 0 for which $A_R(f)$ is <u>not</u> a spider's web.

$m(r)$ small $\Rightarrow M(r)$ grows

Beurling 1933 (Thesis, p.96) f analytic in $\{|z|<r_0\}$

$$E = \{r \in (r_1, r_2) : m(r) \le \mu\}, \qquad 0 \le r_1 < r_2 < r_0, \quad 0 < \mu < M(r_1).$$

Then

$$\log \frac{M(r_2)}{\mu} \ge \frac{1}{2} \exp\left(\frac{1}{2} \int_E \frac{dt}{t}\right) \log \frac{M(r_1)}{\mu}.$$

New idea – if stretching fails we get winding

Nicks + R + S 2018 Let f be real with only real zeros.

1. $\log M(r) = o(r^{\frac{1}{2}}) \Rightarrow$ no unbdd Fatou cmpnts.

2. $\rho < 1 \Rightarrow$ no orbits of unbdd wandering domains.

All functions of order < 1 have the form

$$f(z) = cz^{p_0} \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)^{p_n}, \text{ where } p_n \in \mathbb{N},\ |a_n| \uparrow \infty.$$

Key results from complex analysis

1. If f is real with real zeros and $\rho < 2$, then $\log f$ is conformal in $\{z : \operatorname{Im} z > 0\}$.

Probably a standard fact about the Laguerre-Polya class

2. If f is real with real zeros and $\rho < 2$, and γ is a curve in $\{z : \operatorname{Im} z > 0\}$ that meets $C(s)$ and $C(s^{1+a})$, with

$$\frac{1}{M(s)} \leq |f(z)| \leq M(s), \quad \text{for } z \in \gamma,$$

$a > 1$, s^{a^2} large

then $\exists\, \Gamma \subset \gamma$ s.t. $f(\Gamma)$ winds round 0 at least $\log M(s)$ times.

Proof uses result 1 plus an extremal length argument.

Third key result – needed for $\rho < 1$ case

Using results of Cartwright we prove:

$\boxed{\begin{array}{l}\text{3. If } f \text{ is a t.e.f. with } f(0)=1 \text{ and } \exists\, \alpha \in (0,1) \text{ s.t.} \\ \qquad \log M(r) \le r^{\alpha}, \quad \text{for } r \ge 3R^{1/(1-\alpha)} \\ \qquad \log m(r) \le \frac{1}{2}\log M(r), \quad \text{for } r \in (\frac{1}{4}R, \frac{1}{2}R), \\ \text{then} \\ \quad f \text{ has at least } \frac{1}{28}\log M(R) \text{ zeros in } \{z : cR < |z| < R^{1/(1-\alpha)}\}.\end{array}}$

Roughly speaking, if $\rho < 1$ and $m(r) \le M(r)^{1/2}$ on a long interval then f has many zeros in a larger annulus.

$\boxed{\underset{\sim\sim\sim}{\text{Question}}\ \text{If } \rho < 1, \text{ can there be unbdd wandering domains?}}$