Weighted Dirichlet spaces and Q_p

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Dirichlet type spaces

SETUP

- $\mathbb{D} = \{z : |z| < 1\}$ open unit disk in $\mathbb{C}$.
- $\eta \in M(\overline{\mathbb{D}})$ positive measure on $\overline{\mathbb{D}}$,

write $\eta = \mu + \nu$, $\mu = \eta|_{\mathbb{D}}$, $\nu = \eta|_{\partial\mathbb{D}}$.

$$\begin{aligned}\omega(z) &= \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{z-w}\right| d\mu(w) + \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\nu(\zeta)\\ &:= U_\mu(z) + P_\nu(z).\end{aligned}$$

Weighted Dirichlet space $\mathcal{D}_\omega$ is the space of $f \in Hol(\mathbb{D})$

$$\int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < +\infty.$$

Dirichlet type spaces

- $H^2 = H^2(\mathbb{D})$ the Hardy space, $\mu = \delta_0$, $\nu \equiv 0$

$$\begin{aligned} \|f\|_{H^2}^2 &= |f(0)|^2 + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \\ &\approx |f(0)|^2 + \frac{2}{\pi}\int_{\mathbb{D}} |f'(z)|^2 (1-|z|) dA(z) \end{aligned}$$

- D Dirichlet space, $\mu \equiv 0$, $\nu =$ arclength measure on $\mathbb{T}$

$$\|f\|_D^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z)$$

Dirichlet type spaces

- $D_p = \{f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 (1-|z|)^p dA(z) < \infty\}$
- D_ν when $\mu \equiv 0$ introduced by Stefan Richter (1991) - two isometries.
- D_ω general studied in Habilitation thesis of Alexandru Aleman (1993).

$$D_\omega \subset H^2$$

We are interested in

- Möbius invariant spaces $\mathcal{Q}_p$ in connection with Dirichlet type spaces

Möbius invariant spaces

$\varphi \in \mathrm{Aut}(\mathbb{D}) \Longrightarrow \varphi(z) = e^{i\theta}\sigma_a(z), \quad \sigma_a(z) = \frac{a-z}{1-\bar{a}z}, \quad \theta \in \mathbb{R}, \quad a \in \mathbb{D}$

X a Banach space of analytic functions on $\mathbb{D}$ is called Möbius invariant if

$$f \in X, \varphi \in \mathrm{Aut}(\mathbb{D}) \Longrightarrow f \circ \varphi \in X, \quad \|f \circ \varphi\|_X = \|f\|_X$$

- Bloch space $\mathcal{B}$: $\quad \|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$
- BMOA: analytic functions on $\mathbb{D}$ with boundary values of bdd mean oscillation

$$\|f\|_{BMOA} = |f(0)| + \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2}$$

Möbius invariant spaces

- $\mathcal{Q}_p, 0 \leq p < \infty$: 1995, R. Aulaskari, J. Xiao and R. Zhao,

$\|f\|^2_{\mathcal{Q}_p} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1 - |\sigma_a(z)|^2\right)^p dA(z) < \infty$

$p = 0 \Longrightarrow \mathcal{Q}_0 = D$ Dirichlet space
$p = 1 \Longrightarrow \mathcal{Q}_1 = BMOA$,
$1 < p < \infty \Longrightarrow \mathcal{Q}_p = \mathcal{B}$

Möbius invariant spaces

$(X, \|.\|_X)$ a Banach space of analytic functions in $\mathbb{D}$ containing all constants.

A. Aleman and A. Simbotin: $M(X)$ the Möbius invariant function space generated by X

$$\|f\|_{M(X)} = \sup_{\varphi \in \mathrm{Aut}(\mathbb{D})} \|f \circ \varphi - f(\varphi(0))\|_X < \infty$$

$$X = H^p, 0 < p < \infty \quad \Longrightarrow \quad M(X) = BMOA$$
$$X = A^p, 0 < p < \infty \quad \Longrightarrow \quad M(X) = \mathcal{B}$$
$$X = D_p, 0 < p < 1 \quad \Longrightarrow \quad M(X) = \mathcal{Q}_p$$

SPACE $D_{\mu,p}$

Weighted Green function Let μ be a positive Borel measure on $\mathbb{D}$, $p > 0$,

$$U_{\mu,p}(z) = \int_{\mathbb{D}} (1 - |\sigma_z(w)|^2)^p d\mu(w)$$

$\mathcal{D}_{\mu,p}$: $\quad \|f\|^2_{\mathcal{D}_{\mu,p}} = \int_{\mathbb{D}} |f'(z)|^2 U_{\mu,p}(z) dA(z) < \infty$ (BGP, 2017)

$0 < p \leq 1 \quad \Longrightarrow \quad U_{\mu,p}$ is superharmonic, otherwise not

$\mu = \delta_0 \quad \Longrightarrow \quad \mathcal{D}_{\mu,p} = \mathcal{D}_p$

Möbius invariant spaces

Denote by $\mathbb{F}$ the set of all finite positive Borel measures and by $\mathbb{P}$ the set of all probability measures on $\mathbb{D}$.

Theorem (BGP, 2017)

Let $\mu \in \mathbb{F}$ and $0 < p < \infty$. Then the following are true.

(i) $\mathcal{Q}_p \subsetneq \mathcal{D}_{\mu,p}$.

(ii) $\mathcal{Q}_p = M(\mathcal{D}_{\mu,p})$.

(iii) $\mathcal{Q}_p = \bigcap_{\mu \in \mathbb{P}} \mathcal{D}_{\mu,p}$.

Moreover,

$$\|f\|_{\mathcal{Q}_p} = \sup_{\mu \in \mathbb{P}} \|f\|_{\mathcal{D}_{\mu,p}}$$

Proof of (ii) and (iii)

For (ii), $\mathcal{Q}_p \subset \mathcal{D}_{\mu,p} \subset \mathcal{D}_p$ implies

$$\mathcal{Q}_p = M(\mathcal{Q}_p) \subset M(\mathcal{D}_{\mu,p}) \subset M(\mathcal{D}_p) = \mathcal{Q}_p$$

$$\mathcal{Q}_p \subset \bigcap_{\mu \in \mathbb{P}} \mathcal{D}_{\mu,p} \quad \text{clear!}$$

Suppose $f \notin \mathcal{Q}_p$, that is, $\sup_w \|f\|_{\mathcal{D}_{\delta_w,p}} = \infty$. Choose $w_k \in \mathbb{D}$ so that

$$\beta_k = \|f\|_{\mathcal{D}_{\delta_{w_k},p}} \geq 2^k.$$

Set

$$\nu = \sum_{k=1}^{\infty} 2^{-k} \delta_{w_k}.$$

Then $\nu \in \mathbb{P}$, and $\|f\|^2_{\mathcal{D}_{\nu,p}} = \sum_{k=1}^{\infty} 2^{-k} \beta_k = \infty$. Hence, $f \notin \bigcap_{\mu \in \mathbb{P}} \mathcal{D}_{\mu,p}$.

Composition operators

$\varphi : \mathbb{D} \to \mathbb{D}$ analytic self-map of unit disk $\mathbb{D}$ induces a composition operator

$$C_\varphi f(z) = f(\varphi(z)), \qquad z \in \mathbb{D}, \quad f \in H(\mathbb{D})$$

- Studied like crazy on most known spaces

Nevanlinna counting function of φ with respect to $\mu \geq 0$, $p > 0$

$$N_{\varphi,\mu,p}(z) = \sum_{\varphi(a)=z} U_{\mu,p}(a), \qquad z \in \mathbb{D},$$

multiplicities are taken into account

Change of variables

$$\int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 U_{\mu,p}(z) dA(z) = \int_{\mathbb{D}} |f'(z)|^2 N_{\varphi,\mu,p}(z) dA(z)$$

Subaveraging for $0 < p \leq 1$

$$N_{\varphi,\mu,p}(z) \leq \frac{1}{\text{Area}(\Delta_z)} \int_{\Delta_z} N_{\varphi,\mu,p}(w) dA(w)$$

for any open disk $\Delta_z \subset\subset \mathbb{D}$ with center at z.

Main theorem

Theorem (G, 201?)

Let $p > 1$, $p' > 0$, and let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent.

1. $C_\varphi : \mathcal{B} = \mathcal{Q}_p \to \mathcal{Q}_{p'}$ *is bounded.*
2. $$\sup_{\mu \in \mathbb{P}} \int_{\mathbb{D}} \frac{N_{\varphi,\mu,p'}(z)}{(1-|z|^2)^2} dA(z) < \infty.$$

   $0 < p' \leq 1$
3. *For every $\mu \in \mathbb{P}$, there exists a $\nu \in \mathbb{P}$ such that*

   $$C_\varphi : \mathcal{D}_{\nu,p} \to \mathcal{D}_{\mu,p'}.$$

Main theorem, part 2

Theorem (G, 201?)

Let $p > 1$, $0 < p' \le 1$, and let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent.

1. *$C_\varphi : \mathcal{B} = \mathcal{Q}_p \to \mathcal{Q}_{p'}$ is compact.*
2. *For every $\mu \in \mathbb{P}$,*
$$\lim_{|z|\to 1} \inf_{\nu \in \mathbb{P}} \sup_z \frac{N_{\varphi,\mu,p'}(z)}{U_{\nu,p}(z)} = 0.$$
3. *For every $\mu \in \mathbb{P}$, there exists a $\nu \in \mathbb{P}$ such that*
$$C_\varphi : \mathcal{D}_{\nu,p} \to \mathcal{D}_{\mu,p'}$$
is compact.

Composition operators

Previous work

When $p' = 1$, equivalence of (i) and (ii) in Main Theorem "generalizes" a characterization of bounded/compact $C_\varphi : \mathcal{B} \to BMOA$ by S. Makhmutov and M. Tjani.

$C_\varphi : \mathcal{B} \to \mathcal{B}$ studied by many others:
K. Madigan and A. Matheson; M. Tjani; H. Wulan, D. Zheng and K. Zhu, ...

Composition operators

Theorem (BGP, 2018 when p=1)

Let μ be a positive Borel measure on $\mathbb{D}$, $0 < p \leq 1$, and let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent.

1. C_φ *is bounded on* $\mathcal{D}_{\mu,p}$.
2. $$N_{\varphi,\mu,p}(w) = O(U_{\mu,p}(w)), \qquad \text{as } |w| \to 1.$$
3. $$\frac{1}{A(\Delta_w)} \int_{\Delta_w} N_{\varphi,\mu,p}(z)\,dA(z) = O(U_{\mu,p}(w)), \qquad \text{as } |w| \to 1,$$
   where
   $$\Delta_w = \{z \in \mathbb{D} : \quad |z - w| < \frac{1}{2}(1 - |w|)\}.$$

Composition operators

Theorem (BGP, 2018 when p=1)

Let μ be a positive Borel measure on $\mathbb{D}$, $0 < p \leq 1$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. Then the following conditions are equivalent.

1. C_φ *is compact on* $\mathcal{D}_{\mu,p}$.
2. $$N_{\varphi,\mu,p}(w) = o(U_{\mu,p}(w)), \qquad \text{as } |w| \to 1.$$
3. $$\frac{1}{A(\Delta_w)} \int_{\Delta_w} N_{\varphi,\mu,p}(z)dA(z) = o(U_{\mu,p}(w)), \qquad \text{as } |w| \to 1,$$

   where
   $$\Delta_w = \{z \in \mathbb{D} : \quad |z - w| < \frac{1}{2}(1 - |w|)\}.$$

Composition operators

Theorem (G, 201?)

Let μ be a positive Borel measure on $\mathbb{D}$, $p > 0$, $0 < p' \leq 1$, and let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent.

1. $C_\varphi : \mathcal{D}_{\mu,p} \to \mathcal{Q}_{p'}$ *is bounded.*
2. $$\sup_{\nu \in \mathbb{P}} N_{\varphi,\nu,p'}(w) = O(U_{\mu,p}(w)), \qquad \textit{as } |w| \to 1.$$

In this case, $\|C_\varphi\| \approx \sup_{\frac{1-|\varphi(0)|}{2}<|w|<1} \sup_{\nu \in \mathbb{P}} \frac{N_{\varphi,\nu,p'}(w)}{U_{\mu,p}(w)}$, *and*

$$\|C_\varphi\|_e \approx \limsup_{|w|\to 1} \sup_{\nu \in \mathbb{P}} \frac{N_{\varphi,\nu,p'}(w)}{U_{\mu,p}(w)}$$

Composition operators

Theorem (G, 201?)

Let μ be a positive Borel measure on $\mathbb{D}$, $p > 0$, $0 < p' \leq 1$, and let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent.

1. $C_\varphi : \mathcal{D}_{\mu,p} \to \mathcal{Q}_{p'}$ *is compact.*
2. $$\sup_{\nu \in \mathbb{P}} N_{\varphi,\nu,p'}(w) = o(U_{\mu,p}(w)), \qquad \text{as } |w| \to 1.$$

Open problem

It is currently an open problem to characterize (in terms of function-theoretic properties of φ) bounded or compact composition operators on $\mathcal{Q}_p$ for $0 < p < 1$.

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