Order isomorphisms of countable dense real sets which are universal entire functions

(preliminary report)

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ORDER

An order is a way of giving meaning to an expression of the form

$$x < y.$$

Examples: On people,

Height, weight, age and income are orders.

Nationality, religion, color and gender are not. orders.

ORDER ISOMORPHISMS

Every well-ordered set is order-isomorphic to a unique ordinal. Note: $\mathbb{Q}$ not well-ordered.

Definition. An ordered set is **dense**, if between every two elements, there is a third. Note: $\mathbb{Q}$ is dense.

Cantor 1895

If A and B are countable dense ordered sets without first or last elements, then there is an order isomorphism

$$f : A \to B.$$

Corollary

If A and B are countable dense subsets of $\mathbb{R}$,
then there is an order homeomorphism

$$f : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad f(A) = B.$$

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Corollary (suggested by Solynin talk)

If A and B are countable dense subsets of the circle $\mathbb{T}$,
then there is an order homeomorphism

$$f : \mathbb{T} \to \mathbb{T} \quad \text{with} \quad f(A) = B.$$

Corollary to Cantor 1895
If A and B are countable dense subsets of $\mathbb{R}$,
then there is an order homeomorphism

$$f : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad f(A) = B.$$

Stäckel 1895
If A and B are countable dense subsets of $\mathbb{R}$,
then there is an entire function f

$$f : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad f(A) \subset B.$$

Erdös 1957
A, B countable dense, $\exists$ entire function f with $f(A) = B$?
Yes
$A, B \subset \mathbb{C}$, Maurer 1967. $A, B \subset \mathbb{R}$, Barth-Schneider 1970.

Franklin 1925
If A and B are countable dense subsets of $\mathbb{R}$,
then there is an order bianalytic mapping

$$f : \mathbb{R} \to \mathbb{R}, \quad \text{with} \quad f(A) = B.$$

Corollary to Cantor 1895
If A and B are countable dense subsets of $\mathbb{R}$,
then there is an order homeomorphism

$$f : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad f(A) = B.$$

Corollary (suggested by Solynin talk)
If A and B are countable dense subsets of $\mathbb{T}$,
then there is an order homeomorphism

$$f : \mathbb{T} \to \mathbb{T} \quad \text{with} \quad f(A) = B.$$

Franklin 1925
$A, B \subset \mathbb{R}$ countable dense. Then $\exists$ order bianalytic

$$f : \mathbb{R} \to \mathbb{R}, \quad \text{with} \quad f(A) = B.$$

Question (suggested by Solynin talk)
$A, B \subset \mathbb{T}$ countable dense. Is there is a diffeomorphism

$$f : \mathbb{T} \to \mathbb{T} \quad \text{with} \quad f(A) = B.$$

Franklin 1925 (again)
For A and B countable dense subsets of $\mathbb{R}$, there exists a bianalytic map $f : \mathbb{R} \to f(\mathbb{R}) \subset \mathbb{R}$, such that:
f restricts to a bijection of A onto B (hence, $f(\mathbb{R}) = \mathbb{R}$).

Morayne 1987
If A and B are countable dense subsets of $\mathbb{C}^n$ (respectively $\mathbb{R}^n$), $n > 1,$ there is a measure preserving biholomorphic mapping of $\mathbb{C}^n$ (respectively bianalytic mapping of $\mathbb{R}^n$) which maps A to B.

Rosay-Rudin 1988
Same result for $\mathbb{C}^n$ only.

Remarks

Franklin's proof invokes the statement that the uniform limit of analytic functions is analytic, which is false (in view of Weierstrass approximation theorem, for example).

For $\mathbb{C}^1$, Morayne, Rosay-Rudin results are false.
For $n = 1$, Morayne conclusion $\Rightarrow$ Franklin,
but Morayne proof fails for $n = 1$.

Theorem. For A and B countable dense subsets of $\mathbb{R}$, there exists an entire function f of finite order such that: $f(\mathbb{R}) = \mathbb{R}$; $f'(x) > 0$, for $x \in \mathbb{R}$ and $f|_A : A \to B$ is an order isomorphism .

Proof. $A = \{\alpha_1, \alpha_2, \ldots\}$; $B = \{\beta_1, \beta_2, \ldots\}$.

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \left(z + \sum_{j=1}^{n} \lambda_j h_j(z) \right) = z + \sum_{j=1}^{\infty} \lambda_j h_j(z),$$

$$h_1 = 1; \quad \text{and} \quad h_n(z) = e^{-z^2} \prod_{k=1}^{n-1} (z - \alpha_k), \quad \text{for} \quad n = 2, 3, \ldots,$$

λ_j's small and real $\Rightarrow f(\mathbb{R}) \subset \mathbb{R}$.
λ_j's small $\Rightarrow$ f entire of finite order and $f'(x) > 0, \forall x \in \mathbb{R}$,

$$h_n(z) = 0, \quad \text{iff} \quad z = \alpha_k,\ k = 2, \ldots, n-1.$$

Choose λ_n so $f_n(\alpha_n) = \beta_n$.

$\square$

I OVERSIMPLIFIED

Choose enumerations $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots, \}$. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are rearrangements of $\{a_n\}$ and $\{b_n\}$ chosen recursively.

First, choose $\alpha_1, \lambda_1, \beta_1, \beta_2 \neq \beta_1$, so $f_1(\alpha_1) = \beta_1$.

Suppose we have respectively distinct

$$\alpha_1, \ldots, \alpha_{2n-1}; \quad \lambda_1, \ldots, \lambda_{2n-1}; \quad \beta_1, \ldots, \beta_{2n}$$

$$\alpha_{2k-1} = (\text{first } a_i) \in A \setminus \{\alpha_j : j < 2k-1\}, \quad k = 1, \ldots, n$$

$$\beta_{2k} = (\text{first } b_i) \in B \setminus \{\beta_j : j < 2k\}, \quad k = 1, \ldots, n$$

$$f(\alpha_j) = \beta_j, \quad j = 1, \ldots, 2n-1$$

Choose
$\alpha_{2n}, \lambda_{2n}$,
$\beta_{2n+1}, \alpha_{2n+1}, \lambda_{2n+1}$,
$\beta_{2(n+1)}$
with

$$f_{2n}(\alpha_{2n}) = \beta_{2n} \qquad f_{2n+1}(\alpha_{2n+1}) = \beta_{2n+1}$$

α_1	λ_1	β_1
$-$	$-$	β_2
$-$	$-$	$-$
.	.	.
.	.	.
.	.	.
α_{2n-1}	λ_{2n-1}	β_{2n-1}
$[\alpha_{2n}$	$\lambda_{2n}]$	β_{2n}
α_{2n+1}	$[\lambda_{2n+1}$	$\beta_{2n+1}]$
$-$	$-$	$\beta_{2(n+1)}$

How to find $[\alpha_{2n}, \lambda_{2n}]$ such that

$$\beta_{2n} = f_{2n}(\alpha_{2n}) = \alpha_{2n} + \sum_{j=1}^{2n-1} \lambda_j h_j(\alpha_{2n}) + \lambda_{2n} h_{2n}(\alpha_{2n}) =$$

$$f_{2n-1}(\alpha_{2n}) + \lambda_{2n} h_{2n}(\alpha_{2n}).$$

Put

$$g(x, y) = f_{2n-1}(x) + y h_{2n}(x).$$

Fix y_n small. Show $g(\cdot, y_n) : \mathbb{R} \to \mathbb{R}$ surjective. So, $\exists \alpha$ with $g(\alpha, y_n) = \beta_{2n}$. Implicit function theorem implies, there is $(\alpha_{2n}, \lambda_{2n})$ near (α, y_n), with $g(\alpha_{2n}, \lambda_{2n}) = \beta_{2n}$ and $\alpha_{2n} \in A$.

Universal Functions

Birkhoff 1925 There exists an entire function f which is *universal*. That is, for each entire function g, there is a sequence a_n, such that $f(\cdot + a_n) \to g$.

Most entire functions are universal.

No example is known.

Voronin Universality Theorem 1975
Zero-free holomorphic functions in strip $1/2 < \Re z < 1$ can be approximated by translates of the Riemann zeta-function: $\zeta(z + it_n)$, $t_n \to \infty$.

If the zero-free hyposthesis is superfluous, the Riemann Hypothesis fails.

Bagchi 1981 The following are equivalent:
i) $\exists\, t_n \to \infty$, $\underline{d}\{t_n\} > 0$, $\zeta(\cdot + it_n) \to \zeta$ in strip;
ii) the Riemann Hypothesis is true.

Approximation on Closed Sets

A *chaplet* is a locally finite sequence of disjoint closed discs $\overline{D}_1, \overline{D}_2, \ldots$.

Theorem Given a chaplet $\{\overline{D}_n\}$, a sequence of positive numbers $\{\epsilon_n\}$ and a sequence of functions
$f_n \in A(\overline{D}_n) = C(\overline{D}_n) \cap Hol(D_n)$, there exists
an entire function g, such that, for $n = 1, 2, \ldots,$

$$|g(z) - f_n(z)| < \epsilon_n, \quad \text{for all} \quad z \in \overline{D}_n.$$

Application. The existence of a universal entire function (Birkhoff's Theorem).

Approximation by Functions of Finite Order

With the help of a (not *the*) theorem of Arakelian on approximation by entire functions of finite order, we can prove:

Theorem. For an arbitrary sequence $\epsilon_k > 0$, there exists a sequence $D_k = D(a_k, k)$ such that for every sequence $f_k \in \overline{D}_k$, with $|f_k \epsilon_k| < 1$, there exists an entire function f of finite order, such that

$$|f(z) - f_k(z)| < \epsilon_k, \quad \text{for all} \quad z \in \overline{D}_k.$$

Corollary. For sequences $\mathbb{D}_n = (|z| < n)$, $\varphi_n \in A(\overline{\mathbb{D}_n})$ and $\epsilon_n > 0$, there exists a subsequence a_{k_n} and an entire function f of finite order, such that,
setting $f_n(z) = \varphi_n(z - a_{k_n})$,

$$|f(z + a_{k_n}) - \varphi_n(z)| < \epsilon_{k_n}, \quad \text{for all} \quad z \in \overline{\mathbb{D}}_n.$$

Application (Arakelian). There exist
universal entire functions of finite order.

Given: countable dense real sets A and B,

Theorem (again)
There exists an entire function f of finite order :
f is an order isomorphism of A onto B; $f'(x) > 0,\ x \in \mathbb{R}$;

Can impose other conditions on an order isomorphism f.

Given: increasing sequences a_n and b_n, without limit points,

Theorem (universal-interpolating)
There exists a universal entire function f :
f is an order isomorphism of A onto B;
$f'(x) > 0,\ x \in \mathbb{R}$; and $f(a_n) = b_n,\ n = 1, 2, \dots$

Proof of universal-interpolating theorem .

Lemma. Suppose $\{\overline{D}_n\}$ a chaplet disjoint from $\mathbb{R}$; $\{a_n\}$ and $\{b_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, strictly increasing sequences of real numbers tending to ∞, as $n \to \infty$ and $\epsilon_n > 0$. Then, for every sequence $g_n \in A(\overline{D}_n)$, there exists an entire function Φ, such that $|\Phi - g_n| < \epsilon_n$ on $\overline{D}_n$; Φ maps $\mathbb{R}$ bijectively onto $\mathbb{R}$; $\Phi' > 0$ on $\mathbb{R}$ and $\Phi(a_n) = b_n$, $n = 0, \pm 1, \pm 2, \ldots$.

Lemma. Same hypotheses, there exists entire function H, such that $H_{D_n} \sim 0$, $H_R \sim 1$, $H'_R \sim 0$

Proof of theorem. Replace

$$f(z) = z + \sum_{j=1}^{\infty} \lambda_j e^{-z^2} \prod_{k=1}^{j-1} (z - \alpha_k)$$

by

$$f(z) = \Phi(z) + H(z) \sum_{j=1}^{\infty} \lambda_j e^{-\Phi^2(z)} \prod_{k=1}^{j-1} (\Phi(z) - \Phi(\alpha_k)).$$

Franklin 1925 (again)
For A and B countable dense subsets of $\mathbb{R}$, there exists a bianalytic map $f : \mathbb{R} \to \mathbb{R}$, such that:
f restricts to an order isomorphism of A onto B.

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Remarks

1. (n=1, real case) Franklin's proof incorrect.

2. (n=1, real case) The only measure preserving order isomorphisms of $\mathbb{R}$ are translations $x \mapsto x + c$. The only possible image of a real set A is the real set $B = A + c$.

(n=1, complex case) Barth/Schneider 1972
If A and B are countable dense subsets of $\mathbb{C}$, there exists an entire function f, such that $f(z) \in B$ if and only if $z \in A$.

Paucity

Let $\mathcal{E}$ denote the space of entire functions and $\mathcal{E}_R$ denote the "real" entire functions, that is, the entire functions which map reals to reals.

Remark. $\mathcal{E}_R$ is a closed nowhere dense subset of $\mathcal{E}$.

Let $\mathcal{E}_\rightarrow$ be the space of functions in $\mathcal{E}_R$, whose restrictions to the reals are non-decreasing.

Remark. $\mathcal{E}_\rightarrow$ is a closed nowhere dense subset of $\mathcal{E}_R$.

EFHARISTO!