Cantor bouquets in spiders' webs

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Basic definitions

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function.

The **Fatou set**, $F(f)$, is the set of points for which there is a neighbourhood where the family of iterates is equicontinuous.

The **Julia set**, $J(f)$, is the complement of the Fatou set.

The **escaping set**, $I(f)$, is the set of points that tend to infinity under iteration.

Spiders' webs

Definition

A set $E \subset \mathbb{C}$ is called a *spider's web* if it is connected and there exists a sequence of bounded simply connected domains G_n with $G_n \subset G_{n+1}$ for $n \in \mathbb{N}$, $\partial G_n \subset E$ for $n \in \mathbb{N}$, and $\cup_{n \in \mathbb{N}} G_n = \mathbb{C}$.

Examples of functions of regular growth whose escaping sets (and many of their Julia sets) are spiders' webs (Rippon & Stallard 2012):

- functions of order $\rho < 1/2$, with

$$\rho = \limsup_{r \to \infty} \frac{\log \log \max_{|z|=r} |f(z)|}{\log r};$$

- functions of finite order with Fabry gaps; and
- many functions exhibiting the pits effect.

Cantor bouquets

Definition

Roughly speaking, the Cartesian product of a Cantor set with the closed half-line $[0, \infty)$. The points in the Cantor set are called the *endpoints*, with each the curves being called a *hair*.

Examples of functions that admit Cantor bouquets in their Julia sets:

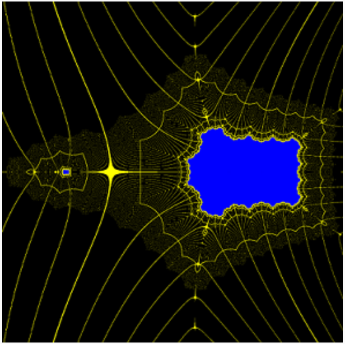
- λe^z for $0 < \lambda < 1/e$, $\mu \sin z$ for $0 < \mu < 1$ (Devaney & Tangerman 1986);
- certain functions with a bounded set of critical and asymptotic values, i.e. in the Eremenko-Lyubich class, (e.g. Barański, Jarque, Rempe 2011); and
- λe^z, $\lambda \in \mathbb{C}^*$ (Bodelón, Devaney, Hayes, Roberts, Goldberg, Hubbard 1999).

Cantor bouquets and spiders' webs

Part of the escaping set of
$$z \mapsto \frac{1}{4}e^z.$$

A Cantor bouquet.

Part of the escaping set of
$$z \mapsto \frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4}).$$



A spider's web.

The case λe^z for $0 < \lambda < 1/e$

Let $E(z) = \lambda e^z$ for some $0 < \lambda < 1/e$.

- E has two fixed points; $0 < q < 1$ is attracting and $p > 1$ is repelling.
- All points z with $\mathrm{Re} < p$ lie in the basin of attraction of q, which is open and dense in $\mathbb{C}$.
- $J(E)$ is the complement of this basin and a Cantor bouquet, consisting of uncountably many, pairwise disjoint curves.

The case λe^z for $0 < \lambda < 1/e$

We can locate a Cantor bouquet in this case as follows.

- For fixed $N \in \mathbb{N}$, define $2N + 1$ horizontal half-strips of width 2π in the right half-plane; $\{T_k : k = -N, \dots, N\}$.
- Let Λ_N be the set of points that stay in $\cup_{|k| \leq N} T_k$ under iteration. The sequence of integers $s = s_0 s_1 \dots$ defined by

  $$E^n(z) \in T_{s_n}$$

  is called the *address* of $z \in \Lambda_N$.
- To each address with $|s_j| \leq N$ for all $j \in \mathbb{N}$, there corresponds a unique curve in Λ_N with the property that each point in this curve shares the same address.

Cantor bouquets in a spider's web

We define the family of transcendental entire functions

$$\mathcal{E} = \cup_{n\geq 3} \left\{ f : f(z) = \sum_{k=0}^{n-1} \exp\left(\omega_n^k z\right) \right\},$$

where $\omega_n = \exp(2\pi i/n)$ is an nth root of unity.

Theorem (Sixsmith 2015)

Let $f \in \mathcal{E}$. Then $I(f)$ and $J(f)$ are spiders' webs of positive area.

We prove the following:

Theorem

Let $f \in \mathcal{E}$. Then there exist Cantor bouquets inside $J(f)$.

Curves are in the Julia set

Lemma (Sixsmith 2015)

Suppose that f is a transcendental entire function and that $z_0 \in I(f)$. Set $z_n = f^n(z_0)$, for $n \in \mathbb{N}$. Suppose that there exist $\lambda > 1$ and $N \geq 0$ such that

$$f(z_n) \neq 0 \quad \text{and} \quad \left| z_n \frac{f'(z_n)}{f(z_n)} \right| \geq \lambda, \quad \text{for } n \geq N.$$

Then either z_0 is in a multiply connected Fatou component of f, or $z_0 \in J(f)$.