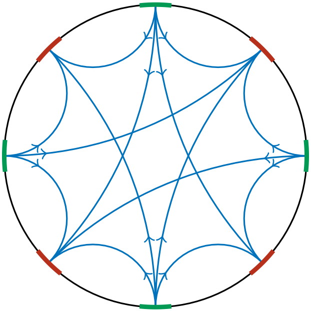
Universal constraints on semigroups of hyperbolic isometries

Argyris Christodoulou

The Open University



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Problem and Motivation

Determine all the finite collections of hyperbolic isometries $f_1, f_2, \ldots, f_n$ for which the semigroup $\langle f_1, f_2, \ldots, f_n \rangle$ satisfies certain discreteness properties.

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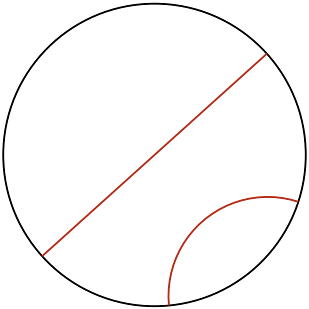
A. Avila, J. Bochi and J.-C. Yoccoz, *Uniformly hyperbolic finite-valued $SL(2, \mathbb{R})$-cocycles*, Comment. Math. Helv. **85** (2010), no. 4, 813–884.

M. Jacques, I. Short, *Dynamics of hyperbolic isometries*, available at https://arxiv.org/abs/1609.00576.

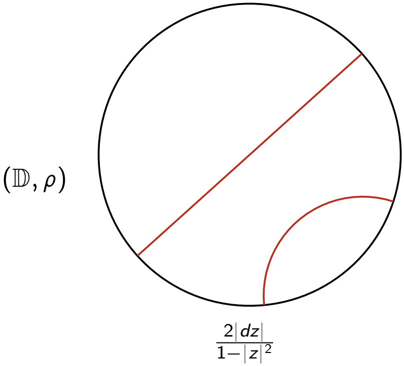
Hyperbolic geometry

$(\mathbb{D}, \rho)$



$\frac{2|dz|}{1-|z|^2}$

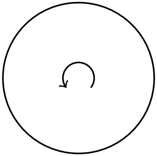
Hyperbolic geometry



Isometries of the hyperbolic plane:

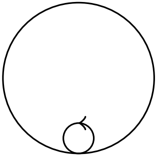
$$z \mapsto e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}, \text{ where } \theta \in \mathbb{R}, z_0 \in \mathbb{D}.$$

Classification of Möbius transformations



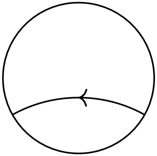
elliptic

one fixed point
inside



parabolic

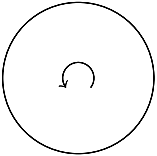
one fixed point
on the boundary



hyperbolic

two fixed points
on the boundary

Classification of Möbius transformations



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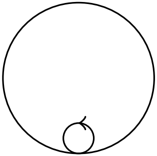
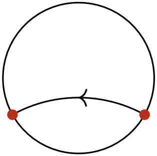


parabolic

one fixed point
on the boundary



hyperbolic

two fixed points
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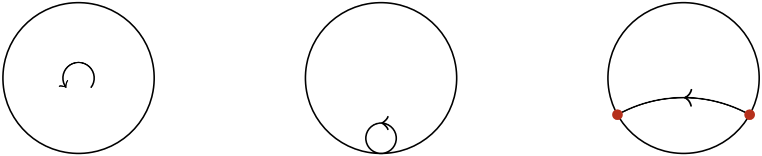

elliptic | parabolic | hyperbolic

one fixed point inside | one fixed point on the boundary | two fixed points on the boundary

Definition

The *translation length* of a hyperbolic transformation f is the distance $\rho(f(w), w)$, for any point w on the axis of f.

Semigroups of Möbius transformations

Definition

In this talk, a *semigroup* is a collection of Möbius transformations that is closed under composition.

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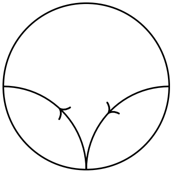
Example. A semidiscrete semigroup that is not discrete.



Finitely-generated Semigroups

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A semigroup S is called *finitely-generated* if there exists a finite collection of Möbius transformations $\mathcal{F}$ such that every element of S can be written as a composition of elements of $\mathcal{F}$.

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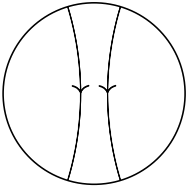
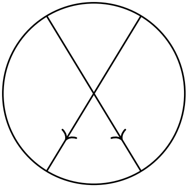
Theorem (Jacques–Short, 2017)

Let S be a finitely-generated semigroup. If there exists a non-trivial closed subset X of $\overline{\mathbb{D}}$ that is mapped strictly inside itself by each generator, then S is semidiscrete.

Examples





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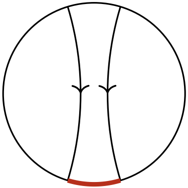

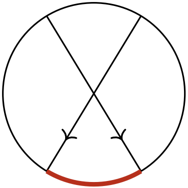

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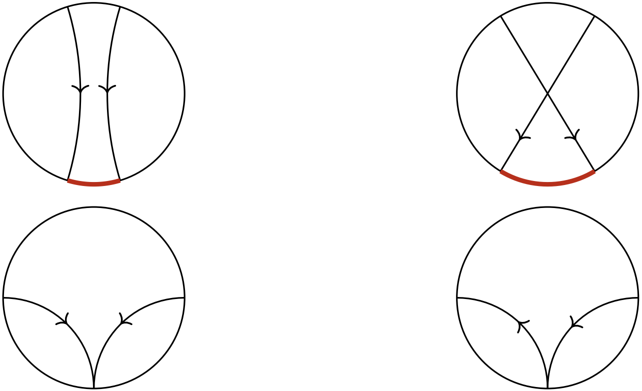

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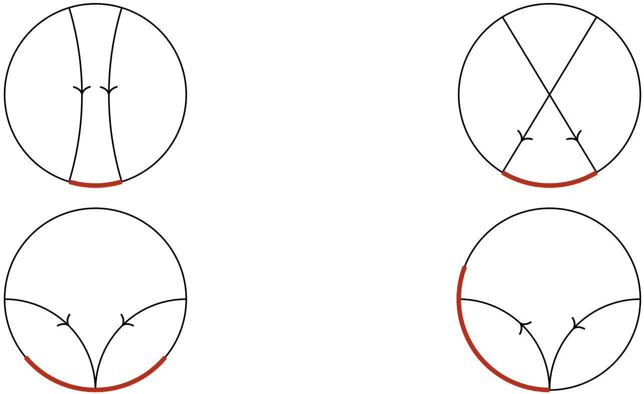

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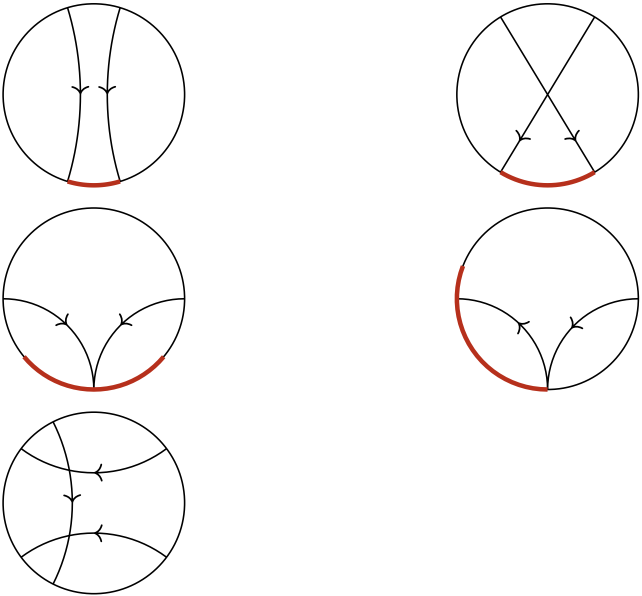

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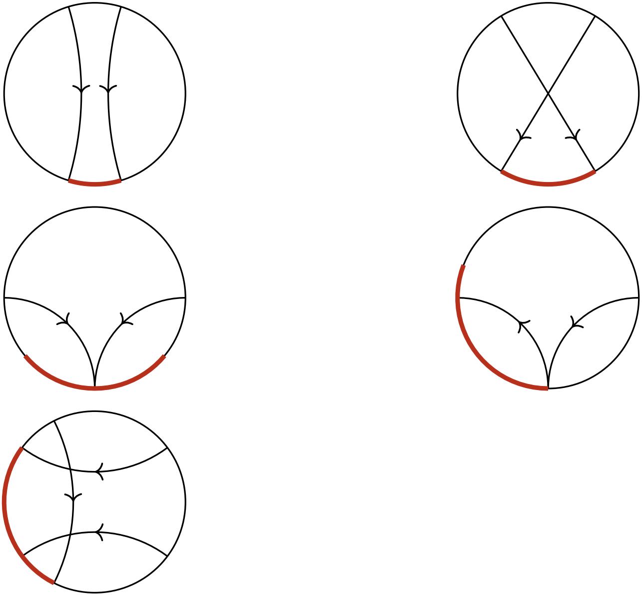

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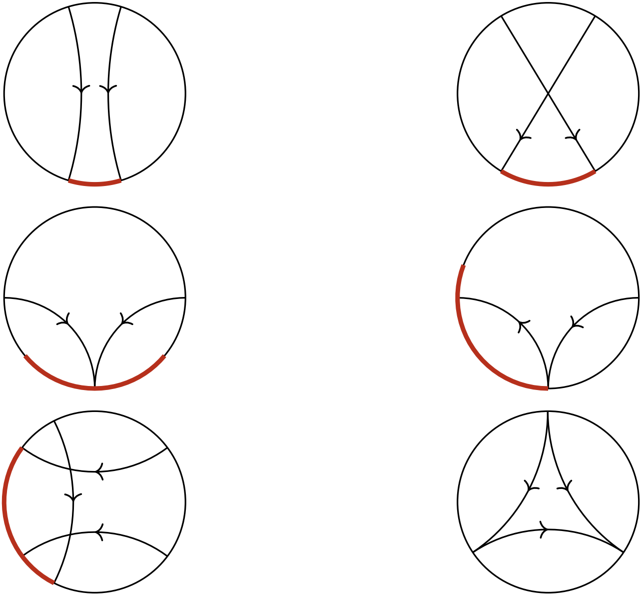

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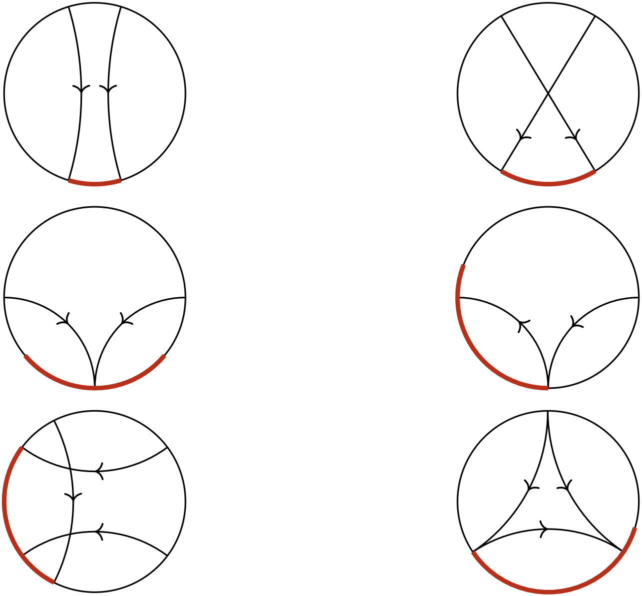

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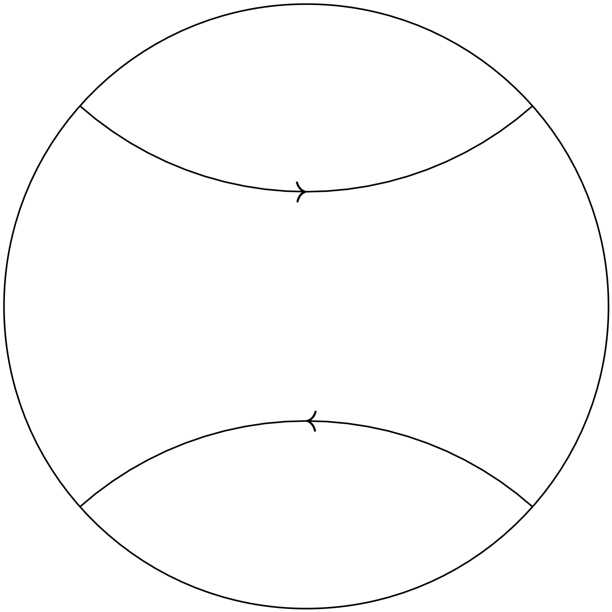

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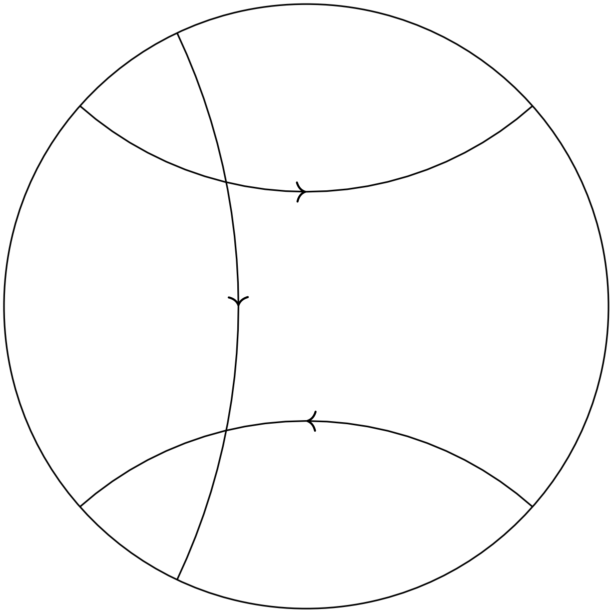

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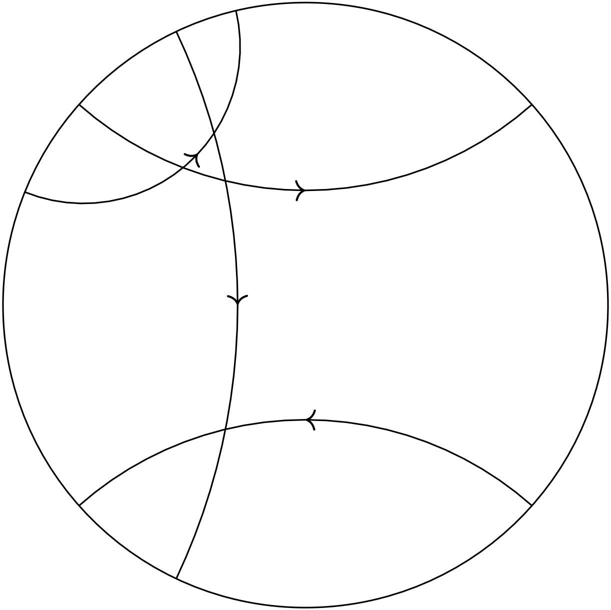

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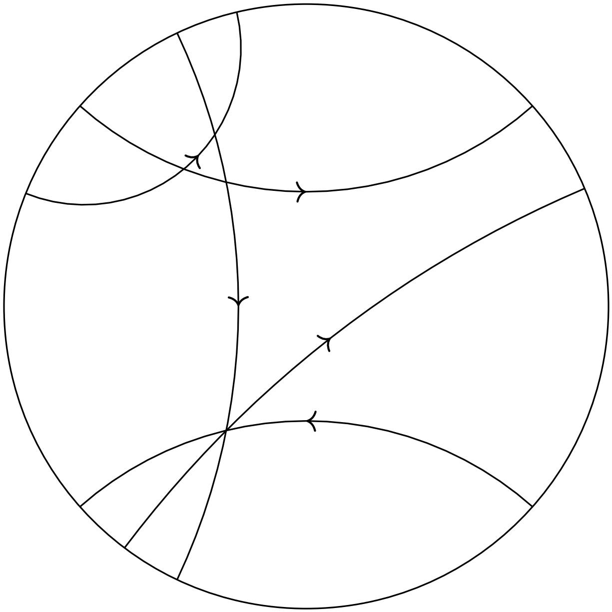
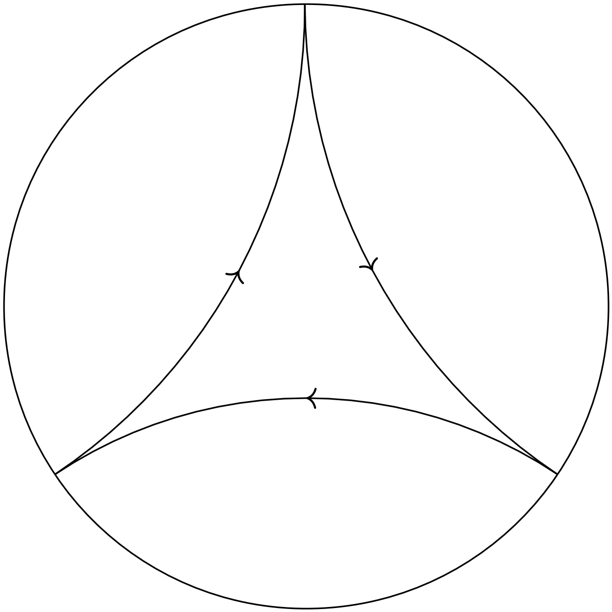


Exceptional cases



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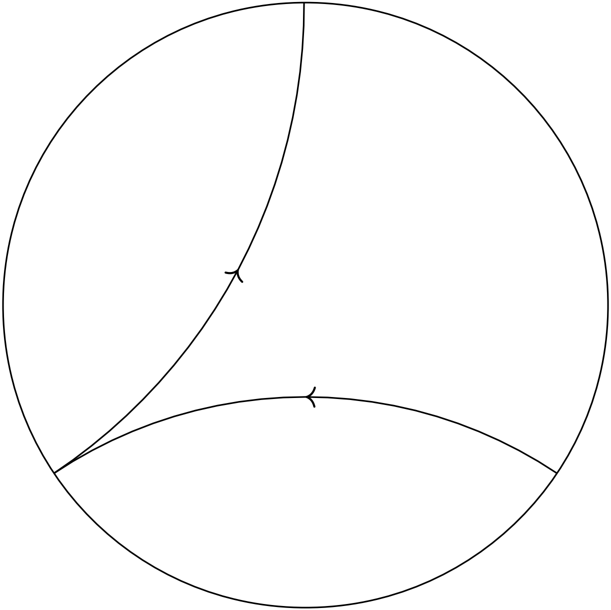

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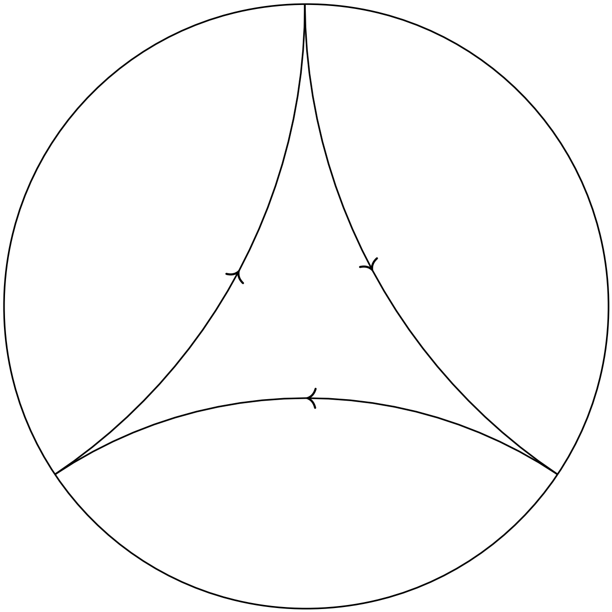

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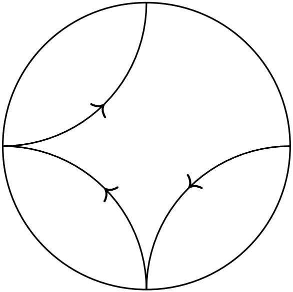

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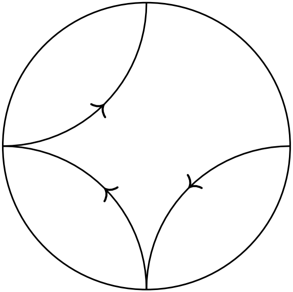

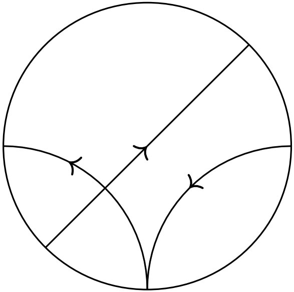

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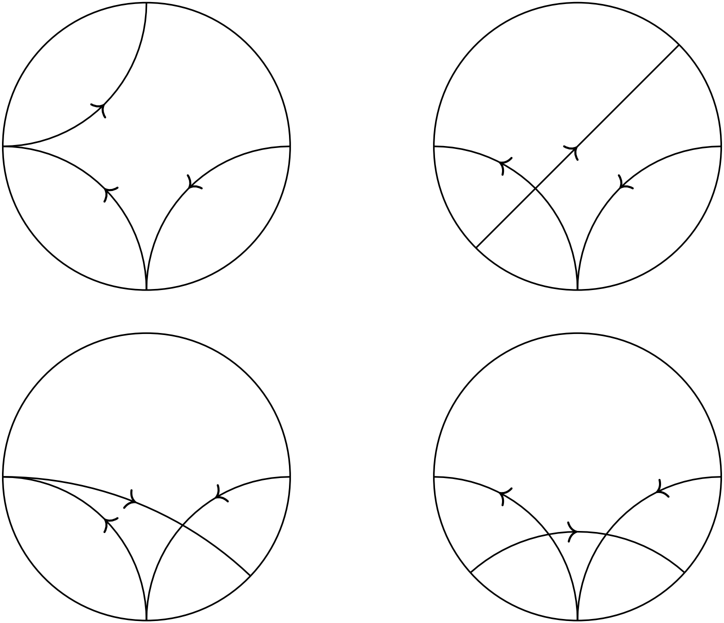

Constraints on the translation lengths

Theorem

Suppose that S is a semigroup generated by the hyperbolic transformations $f_1, f_2, \ldots, f_n$, and let τ_i be the translation length of f_i. There exist $\varepsilon > 0$ and $M > 0$ such that:

(i) *if $\tau_i > M$, for all $i \in \{1, \ldots, n\}$, then S is semidiscrete,*

(ii) *if $\tau_j, \tau_k < \varepsilon$, for some $j, k \in \{1, \ldots, n\}$, then S is not semidiscrete.*

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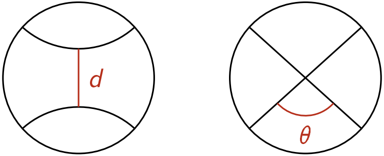
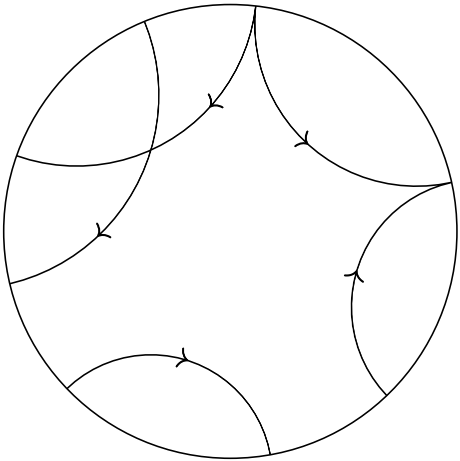
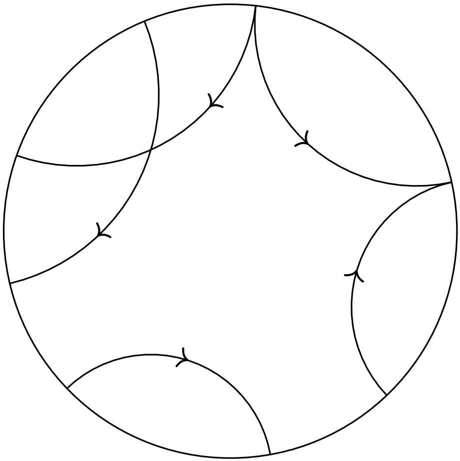
The numbers ε and M depend only on the geometric configuration of the axes of the generators.





Sketch of proof



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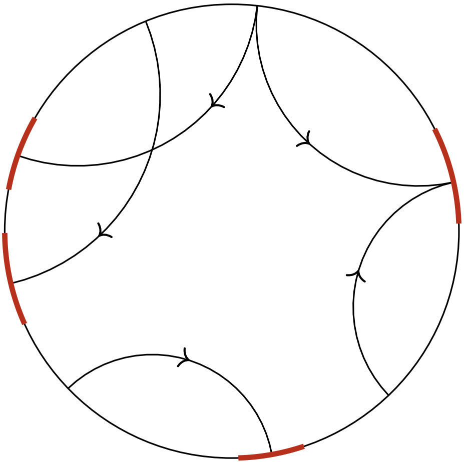

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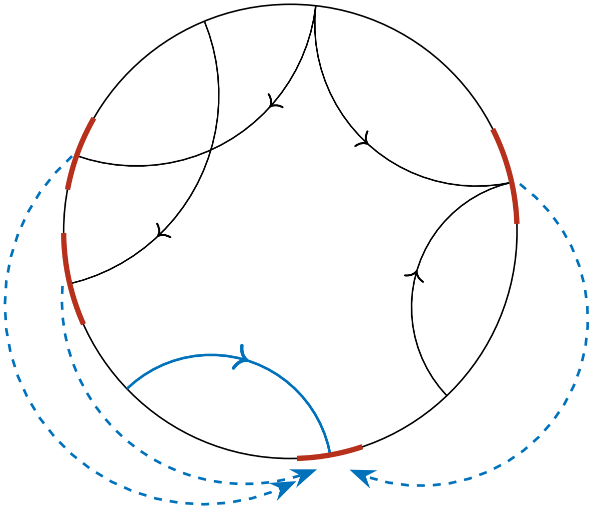

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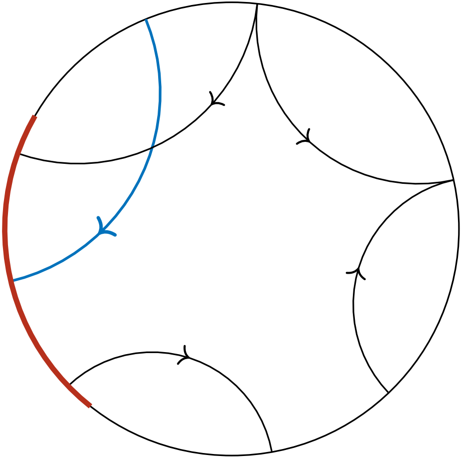

Future work

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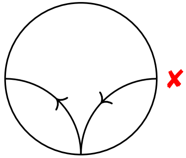

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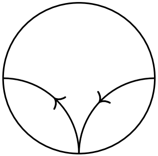

✘ if $\tau_i > M$, for all $i = 1, \ldots, n$, then S is *discrete*.