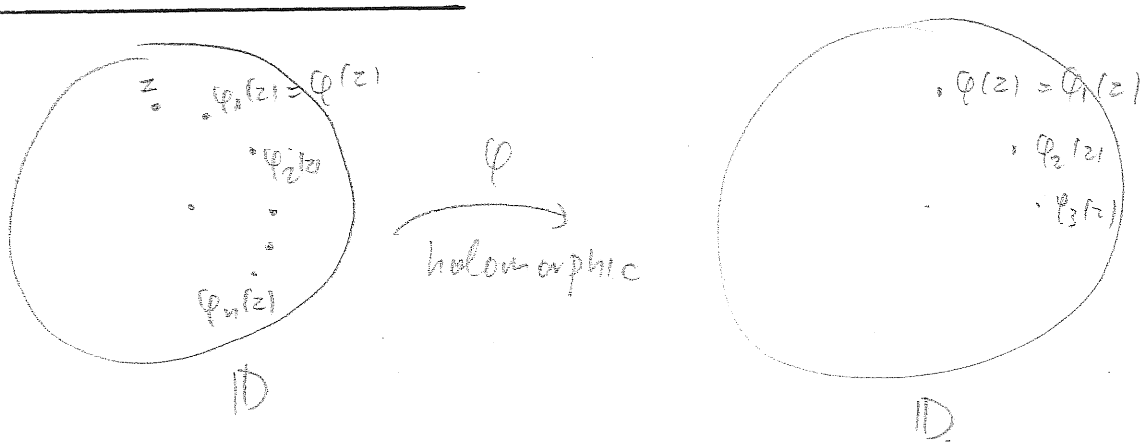
Semigroups of holomorphic self maps on the unit disk

1 Iteration



Let φ be a holomorphic self-map of $\mathbb{D}$.

Consider the sequence of iterates $\varphi_n = \underbrace{\varphi \circ \dots \circ \varphi}_{n \text{ times}}$

Note that $\varphi_n \circ \varphi_m = \varphi_{n+m}$.

We are interested in the limit $\varphi_n \xrightarrow[n\to\infty]{} ?$

Case 1 φ has a fixed point, say $\varphi(0)=0$.

The Schwarz-Pick lemma implies that if φ is not the identity, then φ has a unique fixed point.

Also: If φ is not a rotation, then $|\varphi'(0)|<1$ and $\varphi_n(z) \xrightarrow[n\to\infty]{} 0$, $\forall z \in \mathbb{D}$.

Case 2 Denjoy-Wolff (1926): If φ has no fixed points then $\exists p \in \partial\mathbb{D}$ s.t

(1) $\varphi_n(z) \xrightarrow[n\to\infty]{} p$ $\forall z \in \mathbb{D}$, (2) $\varphi(p) = p$ (angular limit)

(3) $\varphi'(p) \in (0,1]$ angular derivative.

2. Semigroups

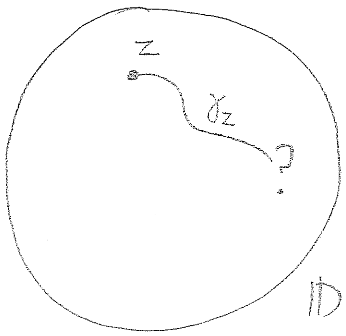
Now we consider "fractional iteration": we substitute the parameter $n \in \mathbb{N}$ with a continuous parameter $t \in [0, +\infty)$.

Definition A family $\{\varphi_t\}$, $t \in [0, +\infty)$ of holomorphic self maps of $\mathbb{D}$ is called <u>semigroup</u> if

(a) $\varphi_0 = id$, (b) $\varphi_{t+s} = \varphi_t \circ \varphi_s$, (c) $\varphi_t(z) \xrightarrow[t \to t_0]{} \varphi_{t_0}(z)$;

$\forall t_0 \in [0, +\infty)$, $\forall z \in \mathbb{D}$

If $z \in \mathbb{D}$ is fixed the path $\gamma_z : [0, +\infty) \to \mathbb{D}$ with $\gamma_z(t) = \varphi_t(z)$ is called <u>the trajectory of z</u>



$\gamma_z(0) = \varphi_0(z) = z$.

The trajectory starts from z.

Proposition 1 If $\varphi_{t_0}(z_0) = z_0$, for some $z_0 \in \mathbb{D}$, $t_0 > 0$, then $\varphi_t(z_0) = z_0$, $\forall t \geq 0$.

<u>Proof</u>

$\forall t > 0$: $\varphi_{t_0}(\varphi_t(z_0)) = \varphi_t(\varphi_{t_0}(z_0)) = \varphi_t(z_0)$.

So $\varphi_t(z_0)$ is fixed point of φ_{t_0} , $\forall t > 0$.

$\Longrightarrow$ $\varphi_t(z_0) = z_0$.

Proposition 2 Each φ_t, $(t \geq 0)$ is univalent 3

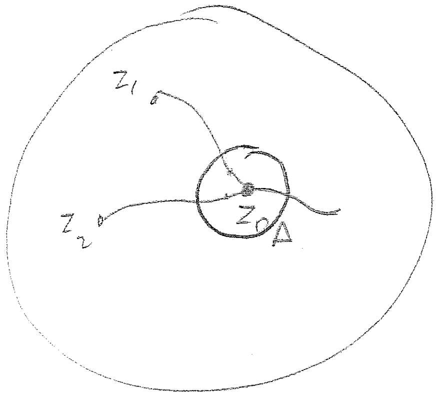
Proof

First observe that $(C) \Rightarrow \varphi_t' \longrightarrow 1$. So for small $t>0$, φ_t is locally univalent.

Suppose that $\varphi_{t_0}(z_1) = \varphi_{t_0}(z_2) = z_0$ for some $t_0 > 0$, $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$.

Let t_0 be the least such t.

$\forall t > t_0$, $\quad \varphi_t(z_{1,2}) = \varphi_{t-t_0}(\varphi_{t_0}(z_{1,2})) = \varphi_{t-t_0}(z_0)$



If two trajectories meet, they stay together. Take a small disk Δ centered at z_0.

For small $t>0$,

$$\varphi_t\Big(\underbrace{\varphi_{t_0-t/2}(z_1, z_2)}_{\in\Delta}\Big) = \varphi_{t/2}(\varphi_{t_0}(z_1, z_2)) = \varphi_{t/2}(z_0)$$

So φ_t is not univalent in Δ: contradiction.

References

1978 − 1985 Berkson-Porta (1978), Heins (1981), Cowen (1981), Siskakis (1985)

2004 - now Contreras, Diaz-Madrigal, Pommerenke, Elin, Shoikhet, Bracci, Gumenyuk. etc.

3. The continuous BW theorem

If φ_t is not an elliptic automorphism of $\mathbb{D}$ then $\exists! p \in \overline{\mathbb{D}}$ s.t $\varphi_t(z) \xrightarrow[t\to\infty]{} p$, $\forall z \in \mathbb{D}$.

Idea of "Proof"

Consider the function $\varphi_1 : \mathbb{D} \to \mathbb{D}$. Then

$\varphi_n = \varphi_1 \circ \dots \circ \varphi_1$ (the iteration sequence of φ_1 is "inside" the semigroup (φ_t))

If $\varphi_1(0) = 0$, then $\varphi_t(z) \xrightarrow[t\to\infty]{} 0$, $\forall z \in \mathbb{D}$

If the attractive point of φ_1 is $p \in \partial\mathbb{D}$, then

$\varphi_t \xrightarrow[t\to\infty]{} p$

4. Classification

If $p \in \mathbb{D}$, then (φ_t) elliptic

If $p \in \partial\mathbb{D}$ and $\varphi_t'(p) < 1$, then (φ_t) hyperbolic

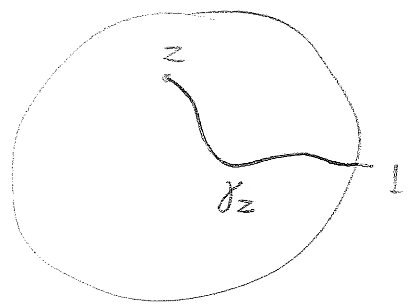
If $p \in \partial\mathbb{D}$ and $\varphi_t'(p) = 1$, then (φ_t) parabolic

Assumption from now on.

$p \in \partial\mathbb{D}$, $p = 1$. the attractive point is 1.

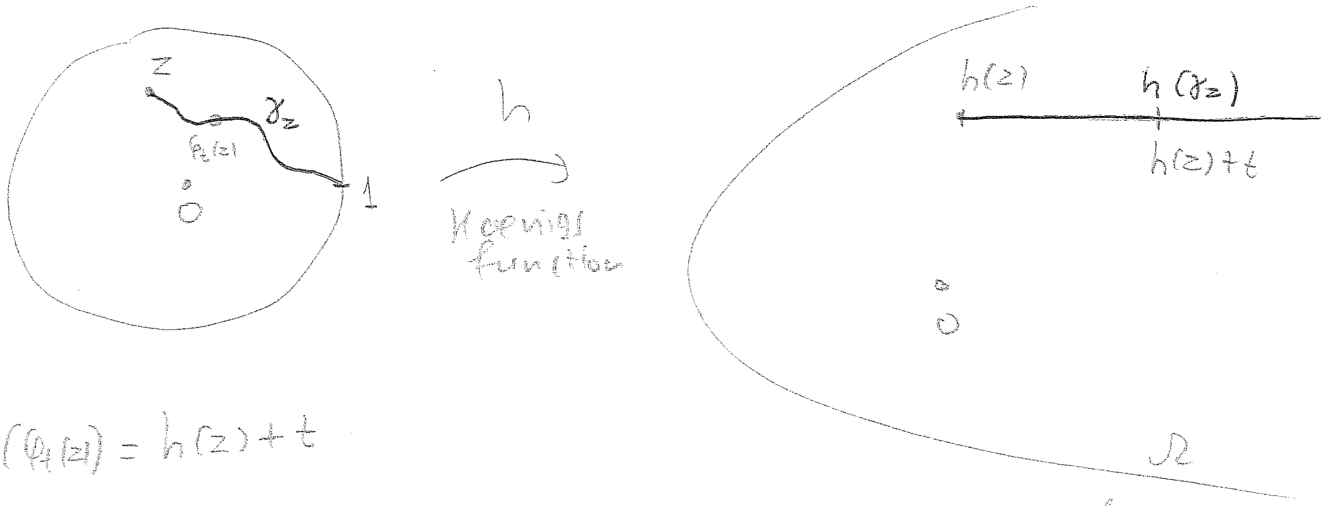
$\varphi_t(1) = 1$, $\forall t \in \mathbb{D}$

$\varphi_t'(1) \in (0,1]$, $\forall t \in \mathbb{D}$.



5. The Koenigs function

Theorem Given a semigroup (φ_t),
$\exists!\, h:\mathbb{D}\to\mathbb{C}$ conformal map with $h(0)=0$
and $\varphi_t(z) = h^{-1}(h(z)+t)$, $\forall t\geq 0$, $\forall z\in\mathbb{D}$.
The domain $\Omega = h(\mathbb{D})$ is <u>positively convex</u>:
$w\in\Omega \Rightarrow w+t\in\Omega$, $\forall t\in[0,+\infty)$



$h(\varphi_t(z)) = h(z)+t$

The trajectories are mapped onto horizontal half-lines.

Conversely, given Ω positively convex, we take the Riemann map $h:\mathbb{D}\xrightarrow{\text{onto}}\Omega$ and define the semigroup $\varphi_t(z) = h^{-1}(h(z)+t)$.

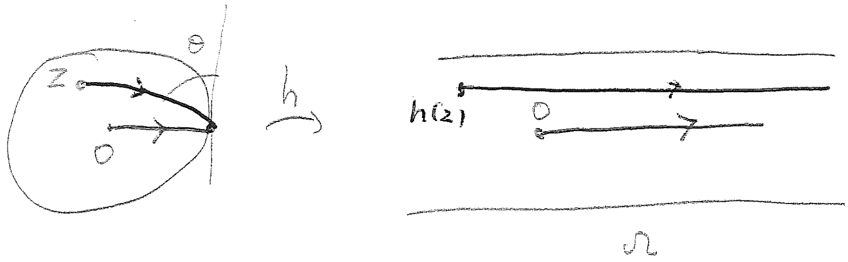
The properties of the semigroup are encoded in the geometry of Ω.

Theorem [2005, CD] (φ_t) is hyperbolic iff $\Omega\subset S$ = horizontal strip.

6 Three basic examples

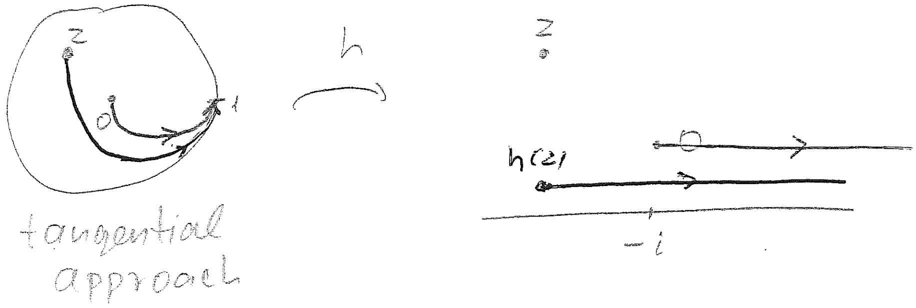
1. Ω = strip

$$h(z) = \log \frac{1+z}{1-z}$$



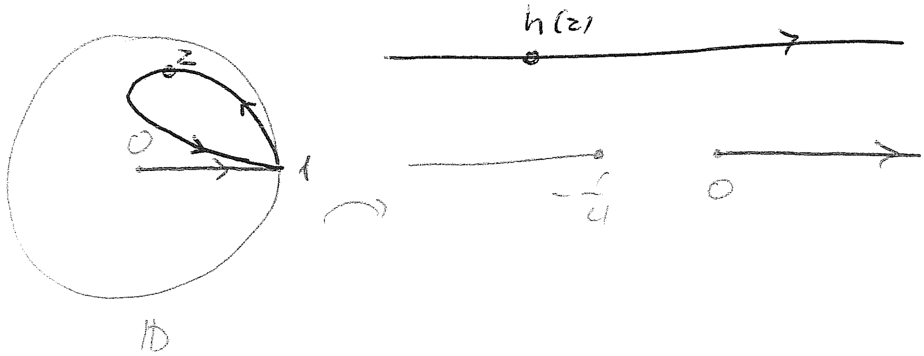
2. Ω = half-plane

$$h(z) = \frac{2zi}{1-z}$$



3. Ω = Koebe

$$h(z) = \frac{z}{(1-z)^2}$$



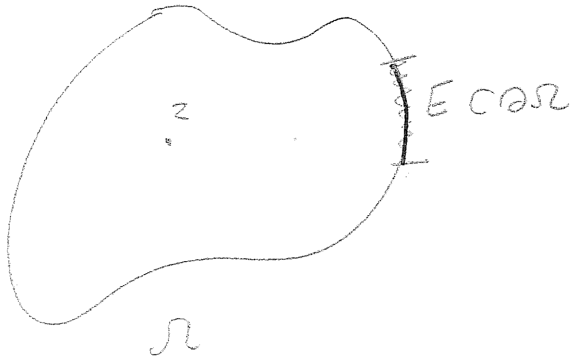
~~7~~. Problem 1 : Rate of convergence

Find a bound of the form $|\varphi_t(z) - 1| \leq ?$

For the three examples, we have

1. Ω = strip $\quad |\varphi_t(z) - 1| \approx e^{-ct}$
2. Ω = half-plane $\quad |\varphi_t(z) - 1| \approx \frac{c}{t}$
3. Ω = Koebe $\quad |\varphi_t(z) - 1| \approx \frac{c}{t^{1/2}}$

8. Harmonic measure



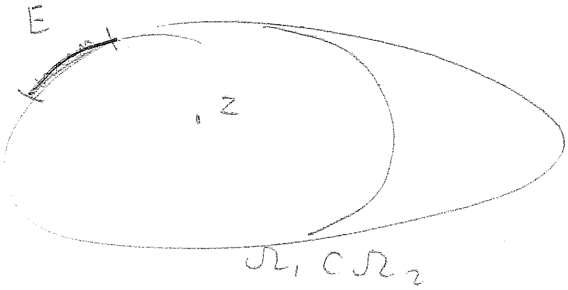
$\Delta u = 0$

$u = 1$ on E

$u = 0$ on $\partial\Omega \setminus E$

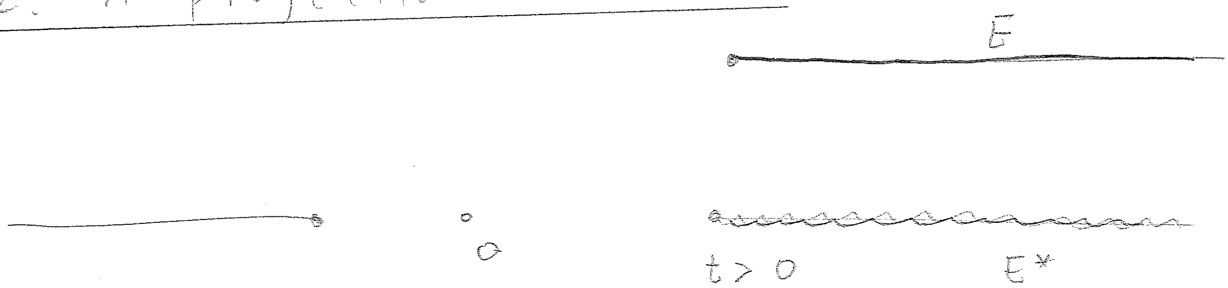
$u(z) = \omega(z, E, \Omega)$

1. Domain monotonicity



$\omega(z, E, \Omega_1) \leq \omega(z, E, \Omega_2)$

2. A projection theorem

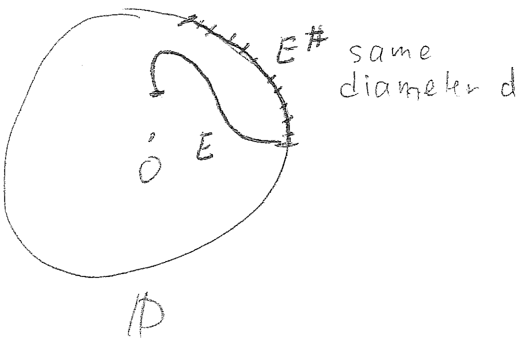


Ω = Koebe domain

$$\omega(0, E, \Omega \setminus E) \leq \omega(0, E^*, \Omega \setminus E^*) \approx \frac{1}{t^{1/2}}$$

(Proof similar to the proof of the BN-projection theorem)

3. A diameter theorem (FRW, Solynin 1985)



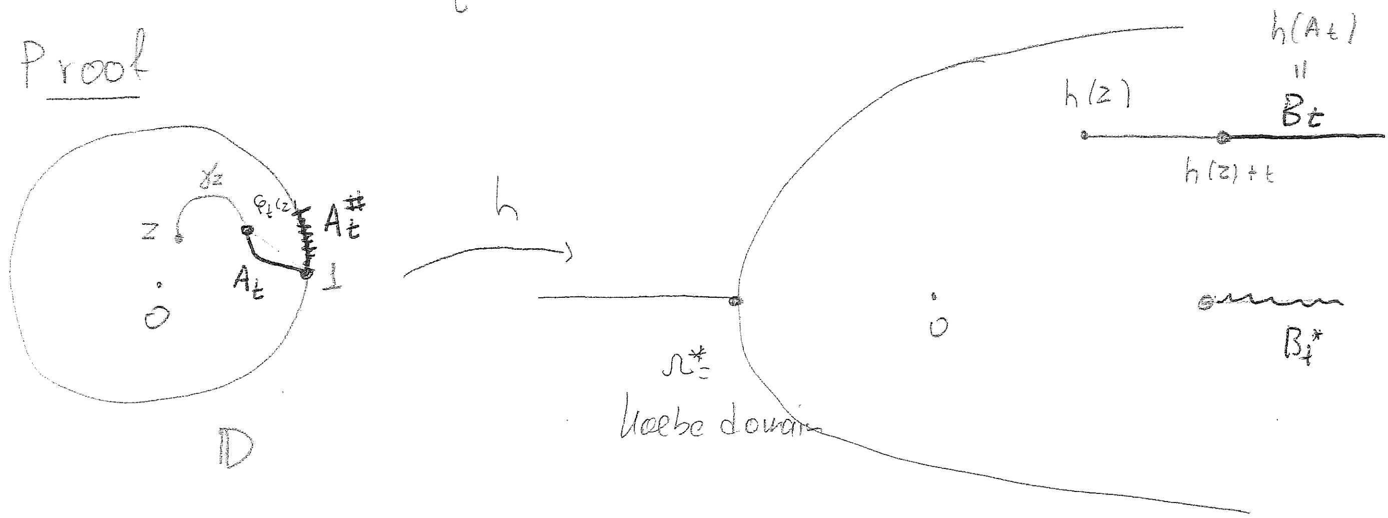
$$\omega(0, E, \mathbb{D} \setminus E) \geq \omega(0, E^\#, \mathbb{D})$$

$$= \frac{1}{\pi} \sin^{-1} \frac{d}{2}.$$

9. **Theorem** For every semigroup,

$$|\varphi_t(z) - 1| \le \frac{C(z)}{t^{1/2}}, \quad \forall z \in \mathbb{D},\ \forall t \in (0, +\infty)$$

Proof



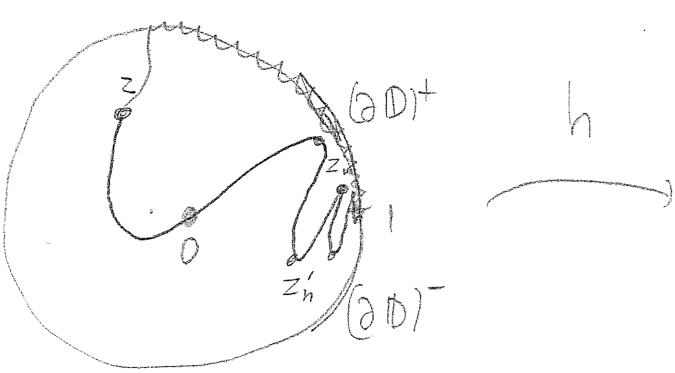
$$\omega(0, A_t, \mathbb{D} \setminus A_t) = \omega(0, B_t, \Omega \setminus B_t)$$
$$\le \omega(0, B_t, \Omega^* \setminus B_t)$$
$$\le \omega(0, B_t^*, \Omega^* \setminus B_t^*)$$
$$= \frac{C}{t^{1/2}}$$

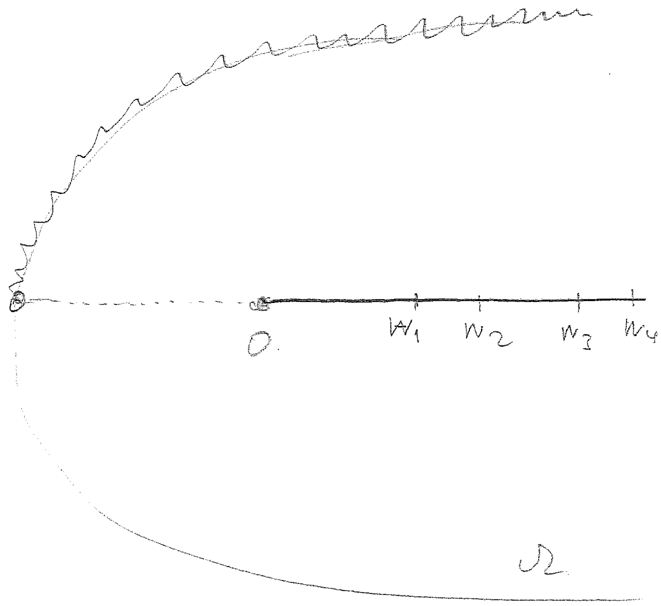
$$\omega(0, A_t, \mathbb{D} \setminus A_t) \ge \omega(0, A_t^\#, \mathbb{D}) = \frac{1}{\pi} \sin^{-1} \frac{d}{2} \ge C_1 d \ge C_1 |1 - \varphi_t(z)|$$

10. Angular behavior of trajectories

If Ω = strip, the trajectories meet the unit circle at the point 1 with a certain angle, nontangentially.
If Ω = half-plane or Ω = Koebe, we have tangential approach.

Question Can we have oscillating trajectories?





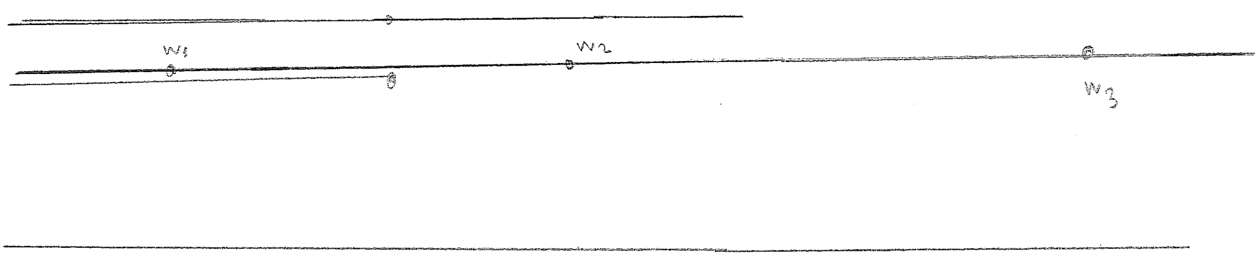
$\omega(z_n, \partial\mathbb{D}^+, \mathbb{D}) \simeq 1$

$\omega(z_n', \partial\mathbb{D}^+, \mathbb{D}) \approx 0$

(tangential oscillation)

Answer: Yes. Here is an example.

Ω



For this domain Ω, take the Riemann map $h: \mathbb{D} \to \Omega$ and define $\varphi_t(z) = h^{-1}(h(z) + t)$.