Polynomial Inequalities in the Complex Plane

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Crete, 2018

Remez-type Inequalities

Remez '36:

$$||p_n||_I \leq T_n\left(\frac{2+s}{2-s}\right) \leq \left(\frac{\sqrt{2}+\sqrt{s}}{\sqrt{2}-\sqrt{s}}\right)^n \leq e^{c\sqrt{s}n}$$

for every real polynomial p_n of degree at most n such that

$$|\{x \in I : |p_n(x)| \leq 1\}| \geq 2-s, \quad 0 < s < 2,$$

where $I := [-1, 1]$ and T_n is the Chebyshev polynomial of degree n.

Set

$$\Pi(p) := \{z \in \mathbb{C} : |p(z)| > 1\}, \quad p \in \mathbb{P}_n.$$

Let now $\Gamma \subset \mathbb{C}$ be an arbitrary bounded Jordan arc or curve.

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For $V \subset \Gamma$ we consider its covering $U = \cup_{j=1}^{m} U_j \supset V$ by a finite number of subarcs U_j of Γ. Set

$$\sigma_\Gamma(V) := \inf \sum_{j=1}^{m} \text{diam } U_j,$$

where the infimum is taken over all finite coverings of V.

Theorem (A. & Ruscheweyh '05). *Let Γ be an arbitrary bounded Jordan arc or curve. If $p \in \mathbb{P}_n$ and*

$$\frac{\sigma_\Gamma(\Gamma \cap \Pi(p))}{\text{diam } \Gamma} =: u < \frac{1}{4},$$

then

$$||p||_\Gamma \le \left(\frac{1 + 2\sqrt{u}}{1 - 2\sqrt{u}}\right)^n \le e^{c\sqrt{u}n}.$$

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Weighted Remez-type Inequalities

Erdélyi '92: Assume that for $p_n \in \mathbb{P}_n$ and $\mathbb{T} := \{z : |z| = 1\}$ we have

$$|\{z \in \mathbb{T} : |p_n(z)| > 1\}| \le s, \quad 0 < s \le \frac{\pi}{2}.$$

Then,

$$||p_n||_{\mathbb{T}} \le e^{2sn}, \quad 0 < s \le \frac{\pi}{2}.$$

A & Ruscheweyh '05:
Let Γ be *quasismooth* (in the sense of Lavrentiev), i.e.,

$$|\Gamma(z_1, z_2)| \le \Lambda_\Gamma |z_1 - z_2|, \quad z_1, z_2 \in \Gamma,$$

where $\Gamma(z_1, z_2)$ is the shorter arc of Γ between z_1 and z_2 and $\Lambda_\Gamma \ge 1$ is a constant.

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Let Ω be the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma$, $\Phi : \Omega \to \mathbb{D}^*$ the Riemann conformal mapping.

For $\delta > 0$, set

$$\Gamma_\delta := \{\zeta \in \Omega : |\Phi(\zeta)| = 1 + \delta\}.$$

Let the function $\delta(t) = \delta(t, \Gamma), t > 0$ be defined by $\operatorname{dist}(\Gamma, \Gamma_{\delta(t)}) = t$.

If for $p_n \in \mathbb{P}_n$,

$$|\{z \in \Gamma : |p_n(z)| > 1\}| \le s < \frac{1}{2} \operatorname{diam} \Gamma,$$

then

$$||p_n||_\Gamma \le \exp(c\delta(s)n)$$

holds with a constant $c = c(\Gamma)$.

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A finite Borel measure ν supported on Γ is an A_∞ *measure* (briefly $\nu \in A_\infty(\Gamma)$) if there exists a constant $\lambda_\nu \geq 1$ such that for any arc $J \subset \Gamma$ and a Borel set $S \subset J$ satisfying $|J| \leq 2|S|$ we have

$$\nu(J) \leq \lambda_\nu \nu(S).$$

The measure defined by the arclength on Γ is the A_∞ measure.

Lavrentiev '36: the equilibrium measure $\mu_\Gamma \in A_\infty(\Gamma)$.

Theorem (A '17) *Let $\nu \in A_\infty(\Gamma)$, $1 \leq p < \infty$, and let $E \subset \Gamma$ be a Borel set. Then for $p_n \in \mathbb{P}_n, n \in \mathbb{N}$, we have*

$$\int_\Gamma |p_n|^p d\nu \leq c_1 \exp(c_2 \delta(s) n) \int_{\Gamma \setminus E} |p_n|^p d\nu$$

provided that $0 < |E| \leq s < (\mathrm{diam}\,\Gamma)/2$, where the constants c_1 and c_2 depend only on Γ, λ_ν, p.

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The sharpness:

Theorem (A '17) *Let* $0 < s < \text{diam}\,\Gamma$ *and* $1 \le p < \infty$*. Then there exist an arc* $E_s \subset \Gamma$ *with* $|E_s| = s$ *as well as constants* $\varepsilon = \varepsilon(\Gamma) > 0$ *and* $n_0 = n_0(s, \Gamma, p) \in \mathbb{N}$ *such that for any* $n > n_0$ *there is a polynomial* $p_{n,s} \in \mathbb{P}_n$ *satisfying*

$$\int_{\Gamma} |p_{n,s}|^p ds \ge \exp(\varepsilon\delta(s)n) \int_{\Gamma\setminus E_s} |p_{n,s}|^p ds.$$

If in the definition of the A_∞ measure we ask S to be also an arc, then ν is called a *doubling measure*. **Mastroianni & Totik '00** constructed an example showing that the weighted Remez-type inequality may not be true in the case of doubling measures.

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Weighted L_p Bernstein-type Inequalities

The starting point of our analysis are the results of **Mastroianni** & **Totik '00** as well as **Mamedkhanov '86, Mamedkhanov** & **Dadashova '09** that extend a classical L_p Bernstein inequality to the case of weighted inequalities for trigonometric polynomials and complex algebraic polynomials over a Jordan curve in the complex plane $\mathbb{C}$.

Let $\Gamma \subset \mathbb{C}$ be a *quasismooth* curve and let Ω be the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma$.
Let ν be a nonnegative Borel measure supported on Γ. We assume that ν satisfies the *doubling condition*

$$\nu(\overline{D(z, 2\delta)}) \leq c_\nu \nu(\overline{D(z, \delta)}), \quad z \in \Gamma, \delta > 0,$$

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Theorem (A '12) *For* $1 \le p < \infty, s \in \mathbb{R}$ *and* $p_n \in \mathbb{P}_n, n \in \mathbb{N}$,

$$\int_\Gamma |p_n'(z)|^p [\rho_{1/n}(z)]^{p+s} d\nu(z)$$

$$\le c(\Gamma, p, c_\nu, s) \int_\Gamma |p_n(z)|^p [\rho_{1/n}(z)]^s d\nu(z).$$

Since the measure $d\nu(z) = |dz|$ satisfies the doubling condition:
Corollary (**Mamedkhanov** & **Dadashova '09**) *Under the assumptions of the above theorem,*

$$\int_\Gamma |p_n'(z)|^p [\rho_{1/n}(z)]^{p+s} |dz| \le c(\Gamma, p, s) \int_\Gamma |p_n(z)|^p [\rho_{1/n}(z)]^s |dz|.$$

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If Γ is Dini-smooth, then

$$\rho_\delta(z) \asymp \delta, \quad z \in \Gamma, \delta > 0.$$

Therefore, in this case

$$\int_\Gamma |p_n'(z)|^p d\nu(z) \le c(\Gamma, p, c_\nu) n^p \int_\Gamma |p_n(z)|^p d\nu(z).$$

Moreover, writing a trigonometric polynomial T_n in the form

$$T_n(x) = e^{-inx} p_{2n}(e^{ix}), \quad p_{2n} \in \mathbb{P}_{2n}$$

and applying the above theorem with $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and $\nu(e^{ix}) = \mu(x)$, we obtain the result of **Mastroianni** & **Totik '00**.

Problem (for trigonometric polynomials **Totik '09**): under which condition on a general (not necessary doubling) measure ν does the weighted Bernstein inequality hold for any $p_n \in \mathbb{P}_n$?
For trigonometric polynomials, see **Bondarenko** & **Tikhonov '15**.

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On the Christoffel Function for the Generalized Jacobi Measures on a Quasidisk

For a finite Borel measure ν on $\mathbb{C}$ such that its support is compact and consists of infinitely many points and a parameter $1 \le p < \infty$, the *n-th Christoffel function* associated with ν and p, is defined by

$$\lambda_n(\nu, p, z) := \inf_{\substack{p_n \in \mathbb{P}_n \\ p_n(z)=1}} \int |p_n|^p d\nu, \quad z \in \mathbb{C}.$$

This function plays an important role in the theory of orthogonal polynomials, in particular, because of the following *Christoffel Variational Principle*

$$\lambda_n(\nu, 2, z) = \left(\sum_{j=0}^{n} |\pi_j(\nu, z)|^2 \right)^{-1}, \quad z \in \mathbb{C},$$

where $\pi_j(\nu, \cdot)$ is the *j-th orthogonal polynomial* with respect to the measure ν.

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Christoffel function

We consider measures supported on the closure $\overline{G}$ of a domain $G \subset \mathbb{C}$ bounded by a Jordan curve $\Gamma := \partial G$. Let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$. The *Riemann mapping function* $\Phi : \Omega \to \mathbb{D}^* := \{w : |w| > 1\}$ normalized by

$$\Phi(\infty) = \infty, \quad \Phi'(\infty) := \lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$$

plays an essential role in our results, which from this point of view, can be compared with recent results in **Totik '10, '14, Varga '13** where the case of a measure ν supported on a Jordan arc or curve is considered

as well as with results in **Suetin '74, Abdullaev '04, Abdullaev & Deger '09, Gustafsson & Putinar & Saff & Stylianopolos '09** where orthogonal polynomials with respect to the weighted area type measures (in particular, Bergman polynomials) are studied.

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Our main attention is paid to the case where G is a bounded *quasidisk*.

For fixed $z_j \in \Gamma := \partial G$ and $\alpha_j > -2, j = 1, \dots, m$, consider the *weight function*

$$h(z) := h_0(z) \prod_{j=1}^{m} |z - z_j|^{\alpha_j}, \quad z \in G,$$

where for a measurable function h_0 the inequality

$$0 < C_h^{-1} \le h_0(z) \le C_h, \quad z \in G$$

holds with a constant $C_h > 1$ depending only on h.

A measure ν supported on $\overline{G}$ and determined by $d\nu = h dm$, where dm stands for the 2-dimensional Lebesgue measure (area) in the plane, is called the *generalized Jacobi measure*.

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$$\Gamma_\delta := \{\zeta \in \Omega : |\Phi(\zeta)| = 1 + \delta\}, \quad \rho_\delta(z) := \text{dist}(\{z\}, \Gamma_\delta).$$

Theorem (A '17) *Let G be a quasidisk, ν be the generalized Jacobi measure, and let $1 \leq p < \infty$. Then for $n \in \mathbb{N} := \{1, 2, \dots\}$ and $z \in \Gamma$,*

$$C^{-1} \leq \lambda(\nu, p, z)\rho_{1/n}(z)^{-2} \prod_{j=1}^{m} (|z - z_j| + \rho_{1/n}(z))^{-\alpha_j} \leq C$$

holds with $C = C(G, h, p) > 1$.

The requirement on G to be a quasidisk cannot be dropped.

The same inequality can be proved if G is replaced by a finite union of quasidisks lying exterior to one other.

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$$C^{-1} \le \lambda(\nu, p, z)\rho_{1/n}(z)^{-2} \prod_{j=1}^{m} (|z - z_j| + \rho_{1/n}(z))^{-\alpha_j} \le C$$

holds with $C = C(G, h, p) > 1$.

The requirement on G to be a quasidisk cannot be dropped.

The same inequality can be proved if G is replaced by a finite union of quasidisks lying exterior to one other.

Christoffel function

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Chebyshev Polynomials

Let $K \subset \mathbb{C}$ be a compact set with $\text{cap}(K) > 0$ and let $T_n(z) = T_n(z, K), n \in \mathbb{N}$ be the n-th *Chebyshev polynomial* associated with K, i.e., $T_n(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_0, c_k \in \mathbb{C}$ is the (unique) monic polynomial which minimizes $||T_n||_K$ among all monic polynomials of the same degree.

Denote by $\widetilde{T}_n$ the n-th Chebyshev polynomial with zeros on K.

It is well-known that

$$||\widetilde{T}_n||_K \geq ||T_n||_K \geq \text{cap}(K)^n,$$

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$$\widetilde{w}_n(K) := \frac{||\widetilde{T}_n||_K}{\text{cap}(K)^n}, \quad w_n(K) := \frac{||T_n||_K}{\text{cap}(K)^n}$$

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Simon ’17: Does the closed domain K bounded by the Koch snowflake obey a *Totik-Widom bound*, i.e.,

$$w_n(K) = O(1) \quad \text{as } n \to \infty?$$

Theorem (A & Nazarov ’18) *Let K be a quasidisk. Then*

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Let now K consist of an infinite number of components.

Carleson '83: a compact set $K \subset \mathbb{R}$ is called *homogeneous* if there is $\eta > 0$ such that for all $x \in K$,

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The classical Cantor set is the uniformly perfect set.

Theorem (A '17) *For a uniformly perfect set $K \subset \mathbb{R}$ there exists $c = c(K) > 0$ such that*

$$w_n(K) = O(n^c) \quad \text{as } n \to \infty.$$

There is a principal difference between the above mentioned classes of compact sets, i.e., K is the Parreau-Widom set in the case of the homogeneous $K \subset \mathbb{R}$ and it is not, in general, the Parreau-Widom set in the case of the uniformly perfect K.

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H. Lebesgue: “I assume that I am not the only one who does not understand the interest in and significance of these strange problems on maxima and minima studied by Chebyshev in memoirs whose titles often begin with “On functions deviating least from zero...". Could it be that one must have a Slavic soul to understand the great Russian Scholar?"

Harmonic majorants in classes of subharmonic functions

Let E_σ be the class of entire functions of exponential type at most $\sigma > 0$.

Bernstein '23: *For $f \in E_\sigma$,*

$$||f'||_{\mathbb{R}} \leq \sigma ||f||_{\mathbb{R}}.$$

Extensions (**Akhiezer '46, Levin '50, '71, '89, Schaeffer '53, Akhiezer & Levin '60, Levin & Logvinenko & Sodin '92**):
If $E \subset \mathbb{R}$ conforms to certain metric properties then for $f \in E_\sigma$,

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We say that a subharmonic function u in $\mathbb{C}$ has degree $\sigma > 0$ if

$$\limsup_{|z|\to\infty} \frac{u(z)}{|z|} = \sigma.$$

Denote by $K_\sigma(E)$ the class of subharmonic in $\mathbb{C}$ functions of degree at most σ and non-positive on E.
Let

$$v(z) = v(z, K_\sigma(E)) := \sup\{u(z) : u \in K_\sigma(E)\}, \quad z \in \mathbb{C}$$

be the *subharmonic majorant* of the class $K_\sigma(E)$. It is known that $v(z)$ is either finite everywhere on $\mathbb{C}$ or equal to $+\infty$ on $\mathbb{C} \setminus E$. The set E is said to be of type (α) in the former case, and of type (β) in the latter.

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Harmonic majorants

Theorem (A '08) *The case (α) holds iff there exist points* $a_j, b_j \in E$, $-\infty < j < \infty$ *such that*

$$b_{j-1} \le a_j < b_j \le a_{j+1}, \quad \lim_{j\to\pm\infty} a_j = \pm\infty,$$

$$\bigcup_{j=-\infty}^{\infty} (a_j, b_j) \supset E^* := \mathbb{R} \setminus E,$$

$$\inf_{-\infty<j<\infty} \frac{\mathrm{cap}(E \cap [a_j, b_j])}{\mathrm{cap}([a_j, b_j])} > 0,$$

$$\sum_{j=-\infty}^{\infty} \left(\frac{b_j - a_j}{|a_j| + 1} \right)^2 < \infty.$$

see also **Carleson** & **Totik '04, Carroll** & **Gardiner '08.**

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Corollary (Schaeffer '53, Benidicks '80, Segawa '88, '90, Levin '89, Gardiner '90). *Since*

$$\mathrm{cap}([a_j, b_j]) = \frac{b_j - a_j}{4} \quad \text{and} \quad \mathrm{cap}(E \cap [a_j, b_j]) \geq \frac{|E \cap [a_j, b_j]|}{4},$$

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is sufficient for the case (α).

Bernstein-type approximation theorem

For a closed unbounded set $E \subset \mathbb{C}$, denote by $BC(E)$ the class of (complex-valued) functions which are bounded and continuous on E. Let E_σ be the class of entire functions of exponential type at most $\sigma > 0$ and let

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Bernstein '46: *for* $f \in BC(\mathbb{R})$ *and* $0 < \alpha < 1$*,*

$$A_\sigma(f, \mathbb{R}) = O(\sigma^{-\alpha}) \quad \text{as } \sigma \to \infty$$

iff

$$\omega_{f,\mathbb{R}}(\delta) = O(\delta^\alpha) \quad \text{as } \delta \to +0,$$

where

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Consider the following two problems:
(a) find the structure properties of $f \in BC(E)$ satisfying

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(we focus on this interpretation of the Bernstein result);

(b) describe the rate of approximation of $f \in BC(E)$ satisfying

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(**Brudnyi '60, Shirokov '03, '04, Shirokov** & **Silvanovich '06, '08, '16, '17**).

The set $E^* := \mathbb{R} \setminus E$ consists of a finite or infinite number of disjoint open intervals $J_j = (a_j, b_j)$. We assume that if the number of J_js is infinite then E possesses the following two properties:

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Example. Let $E = \bigcup_{l=-\infty}^{\infty} [c_l, d_l]$, where

$$d_{l-1} < c_l < d_l < c_{l+1}, \quad l = 0, \pm 1, \pm 2, \dots$$

are such that

$$d_l - c_l \geq C_3, \quad c_{l+1} - d_l \leq C_4.$$

In the case of polynomial approximation of continuous functions on a finite interval $[a, b] \subset \mathbb{R}$, the special role of the endpoints a and b is well-known.
Ditzian & Totik '87: a new modulus of continuity by using the distance between the points on $[a, b]$ that is not Euclidean.

In the case of entire function approximation on E the endpoints of J_j also play a special role.

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In the case of entire function approximation on E the endpoints of J_j also play a special role.

Bernstein-type approximation theorem

Example. Let $E = \bigcup_{l=-\infty}^{\infty} [c_l, d_l]$, where

$$d_{l-1} < c_l < d_l < c_{l+1}, \quad l = 0, \pm 1, \pm 2, \dots$$

are such that

$$d_l - c_l \geq C_3, \quad c_{l+1} - d_l \leq C_4.$$

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In the case of entire function approximation on E the endpoints of J_j also play a special role.

Bernstein-type approximation theorem

Let $\mathbb{H} := \{z : \Im z > 0\}$.
Levin '89: there exist vertical intervals $J_j' = (u_j, u_j + iv_j]$, $u_j \in \mathbb{R}, v_j > 0$ and a conformal mapping

$$\phi : \mathbb{H} \to \mathbb{H}_E := \mathbb{H} \setminus (\cup_j J_j')$$

normalized by $\phi(\infty) = \infty, \phi(i) = i$ such that ϕ can be extended continuously to $\overline{\mathbb{H}}$ and it satisfies the boundary correspondence $\phi(J_j) = J_j'$.

For $x_1, x_2 \in E$ such that $x_1 < x_2$ set

$$\tau_E(x_1, x_2) = \tau_E(x_2, x_1) := \text{diam}\, \phi([x_1, x_2]).$$

In spite of its definition via the conformal mapping, the behavior of τ_E can be characterized in purely geometrical terms. In particular,

$$\tau_E(x_1, x_2) \geq C_5 |x_2 - x_1|, \quad x_1, x_2 \in E.$$

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Bernstein-type approximation theorem

Theorem (A '10) *For* $f \in BC(E)$ *and* $0 < \alpha < 1$,

$$A_\sigma(f, E) = O(\sigma^{-\alpha}) \quad \text{as } \sigma \to \infty$$

iff

$$\omega^*_{f,E}(\delta) = O(\delta^\alpha) \quad \text{as } \delta \to +0,$$

where

$$\omega^*_{f,E}(\delta) := \sup_{\substack{x_1, x_2 \in E \\ \tau_E(x_1, x_2) \le \delta}} |f(x_2) - f(x_1)|, \quad \delta > 0.$$